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Fuzzy portfolio optimization under downside risk measures☆

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Abstract

This paper presents two fuzzy portfolio selection models where the objective is to minimize the downside risk constrained by a given expected return. We assume that the rates of returns on securities are approximated as *LR*-fuzzy numbers of the same shape, and that the expected return and risk are evaluated by interval-valued means. We establish the relationship between those mean-interval definitions for a given fuzzy portfolio by using suitable ordering relations. Finally, we formulate the portfolio selection problem as a linear program when the returns on the assets are of trapezoidal form.

Keywords: Fuzzy numbers; Interval-valued expectation; Possibilistic mean value; Downside risk functions; Fuzzy mathematical programming

1. Introduction

The portfolio selection problem deals with how to form a satisfying share portfolio. It is difficult to decide which securities should be selected because of the existence of uncertainty on their returns. Our main objective is to obtain the optimal proportions for creating a portfolio which respects the investor's declared preferences. It is assumed that the investors wish to strike a balance between maximizing the return and minimizing the risk of their investment.

The first mathematical formulation of the problem of selecting a portfolio in the framework of risk-return trade-off was provided by Markowitz [25], who combines probability theory and optimization theory to model the behaviour of the economic agents. In general portfolio selection problems a probability distribution of the return on the assets is assumed to be known, the return is quantified by means of its expected value and the variance of the portfolio return is regarded as the risk of the investment. This classical mean–variance (MV) model is valid if the return is multivariate-normally distributed and the investor is averse to risk and always prefers lower risk, or it is valid if for any given return which is multivariate distributed, the investor has a quadratic utility function [15]. In contrast to the quadratic Markowitz model, Konno and Yamazaki [19] proposed the first linear model for portfolio selection, the L_1 risk model. This model uses the mean-absolute deviation around the averages as a measure of the risk of the investment and it essentially gives the same results as the MV model when the assets are multivariate-normally distributed. Some authors have recently proposed using downside risk as a measure of the risk of the investment by means of standard

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approaches to the portfolio selection problem [9,24]. Lien and Tse [24] compare the hedging effectiveness of currency futures with respect to currency options on the basis of the lower partial moments, as opposed to the two-sided risk measure.

Notice that a downside risk measure would also help investors make proper decisions when the returns are non-normally distributed, as is the case in emerging market data and for international portfolio selection [3,31].

Fuzzy Set Theory has been widely used to solve many practical problems, including financial risk management, since it allows us to describe and treat imprecise and uncertain elements present in a decision-making problem. Then the imperfect knowledge of the returns on the assets and the uncertainty involved in the behaviour of financial markets may also be introduced by means of fuzzy quantities and/or fuzzy constraints. Different elements can be fuzzified in the portfolio selection problem. Some authors use possibility distributions to model the uncertainty on returns [5,14,17,32,33], while other authors study the portfolio selection problem using fuzzy formulations [1,22,34–36].

In our approach, the uncertainty of the returns on the assets is modelled by means of fuzzy quantities; hence different definitions of the average of a fuzzy number can be used to evaluate both the expected return and the risk of a given portfolio. Dubois and Prade [8] introduce the mean interval of a fuzzy number as a closed interval bounded by the expectations calculated from its lower and upper probability mean values. Alternatively, Carlsson and Fullér [4] define an interval-valued possibilistic mean of fuzzy numbers, their definition being consistent with the extension principle and also based on the set of level-cuts. Weighted mean values introduced in [10] are a generalization of those possibilistic ones and allow us to incorporate the importance of α-level sets. It shows that for *LR*-fuzzy numbers, any f-weighted interval-valued possibilistic mean value is a subset of the interval-valued mean of a fuzzy number in the sense of Dubois and Prade.

Our goal is to present a fuzzy downside risk approach for managing portfolio selection problems in the framework of risk-return trade-off using interval-valued expectations. Section 2 is devoted to describing the relationship between those two interval-valued means for portfolios built using fuzzy returns which have been modelled with *LR*-fuzzy numbers of the same shape. The development of its corresponding fuzzy downside risk functions is given in Section 3. Then in Section 4 we present the formulation of the fuzzy portfolio models which measure the risk of the investment by means of downside functions. We illustrate our approach to selecting the optimal portfolio using numerical examples in Section 5, where a comparison with other selection strategies is also shown.

2. Fuzzy expected return

Some authors use fuzzy numbers to represent the uncertainty of the future return on assets and they set out portfolio selection as a problem of mathematical programming in order to select the best alternative in the decision-making problem [5,21,26]. In this paper we use fuzzy quantities to represent the returns on the financial securities and suitable definitions of their expected returns.

Let us denote by $\tilde{R}_j$ the fuzzy return on the asset jth, its interval-valued mean [8] is defined as the following interval:

$$E(\tilde{R}_j) = [E_*(\tilde{R}_j), E^*(\tilde{R}_j)],$$

where $E_*(\tilde{R}_j) = \int_0^1 (\inf \tilde{R}_{j\alpha})\,\mathrm{d}\alpha$, $E^*(\tilde{R}_j) = \int_0^1 (\sup \tilde{R}_{j\alpha})\,\mathrm{d}\alpha$ and $\inf \tilde{R}_{j\alpha}$ and $\sup \tilde{R}_{j\alpha}$ denote, respectively, the left and right extreme points of the α-level cut of $\tilde{R}_j$ for $\alpha \in (0, 1]$.

It is known that the mean-interval definition in the sense of Dubois and Prade, $E(\tilde{R}_j)$, provides the nearest interval approximation of the fuzzy number $\tilde{R}_j$ with respect to the metric introduced in [12] but not with respect to the Hamming distance [6].

On the other hand, by using the definition of the interval-valued possibilistic mean in [4] for $\tilde{R}_j$, we will have

$$M(\tilde{R}_j) = [M_*(\tilde{R}_j), M^*(\tilde{R}_j)],$$

where $M_*(\tilde{R}_j) = 2\int_0^1 \alpha(\inf \tilde{R}_{j\alpha})\,\mathrm{d}\alpha$ and $M^*(\tilde{R}_j) = 2\int_0^1 \alpha(\sup \tilde{R}_{j\alpha})\,\mathrm{d}\alpha$. And for *LR*-fuzzy numbers with strictly decreasing reference functions, the following inclusion holds: $M(\tilde{R}_j) \subseteq E(\tilde{R}_j)$.

In a standard formulation of the portfolio selection, an investor chooses x_j, the proportion of a total investment fund devoted to asset jth, for n risky assets, so the portfolio may be denoted by $P(x) = \{x_1, \ldots, x_n\}$. In this paper we will establish that for a given portfolio $P(x)$, the above mean-interval expectations may be considered more or less