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Two-sided coherent risk measures and their application in realistic portfolio optimization ☆

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ABSTRACT

By using a different derivation scheme, a new class of two-sided coherent risk measures is constructed in this paper. Different from existing coherent risk measures, both positive and negative deviations from the expected return are considered in the new measure simultaneously but differently. This innovation makes it easy to reasonably describe and control the asymmetry and fat-tail characteristics of the loss distribution and to properly reflect the investor's risk attitude. With its easy computation of the new risk measure, a realistic portfolio selection model is established by taking into account typical market frictions such as taxes, transaction costs, and value constraints. Empirical results demonstrate that our new portfolio selection model can not only suitably reflect the impact of different trading constraints, but find more robust optimal portfolios, which are better than the optimal portfolio obtained under the conditional value-at-risk measure in terms of diversification and typical performance ratios.

1. Introduction

One of the basic problems of applied finance is the optimal selection of stocks, with the conflicting objectives of maximizing future return and minimizing investment risk. The first systematic treatment of this dilemma is the mean–variance (MV) approach proposed by Markowitz (1952). This approach has limited generality since the MV model will only lead to optimal decisions if utility functions are quadratic or if investment returns are jointly elliptically distributed. Quadratic utility is unlikely because it implies increasing absolute risk aversion over the whole domain. The assumption of elliptically distributed returns sounds unrealistic since it rules out possible asymmetry in return distributions. It is nowadays a stylised fact that the distributions of many financial return series are non-normal, with skewness and/or leptokurtosis ('fat tails') pervasive. The well-accepted method for measuring future return is to use the mean of the random return distribution. Nonetheless, the best way to measure the investment risk has not been found even after many different measures having been proposed. Aimed at characterizing an investor's preferences, it is recently examined in Rachev et al. (2008) about desirable properties of an ideal risk measure.

The MV model treats both the above and the below expected returns equally. Nevertheless, there is ample evidence that agents often treat losses and gains asymmetrically (see, for example, Kahneman et al., 1990). The concept of the downside risk apparently has considerable impact on the investor's viewpoint regarding the risk. A general family of below-target risk measures, named the lower partial moment (LPM), was proposed by Bawa (1975) and Fishburn (1977). A comprehensive accumulation of knowledge on the concept of downside risk can be found in Sortino and Satchell (2001). Especially, in the last 10 years, there has been a great momentum in research on quantile-based risk measures because of the introduction of the value-at-risk (VaR) (Morgan, 1996). Unfortunately, VaR has undesirable properties such as lack of sub-additivity; when calculated using scenarios, VaR is non-convex, non-smooth as a function of investment positions and is difficult to optimize (Mausser and Rosen, 1991).

Aimed at providing a comprehensive theory that can relate and compare different risk measure approaches, an important step toward consistent measures of risk was made in a series of papers

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(Artzner et al., 1997, 1999; Delbaen, 2002), which introduced the notion of a coherent risk measure. It is easy to demonstrate that VaR does not provide coherency, neither do existing LPM measures. Extensive researches have been aroused since the appearance of this new definition. Typical measures of this class include the expected shortfall (ES) defined in Acerbi and Tasche (2002) and the conditional value-at-risk (CVaR) developed in Rockafellar and Uryasev (2002). The most general theoretical result about this kind of measures is a complete space of coherent risk measures spectrally generated in Acerbi (2002). The relationship among moment-based, distortion and spectral risk measures is recently investigated in Adam et al. (in press).

These new risk measures alleviate the problems inherent in VaR by considering losses beyond the VaR level. Nevertheless, considered in these measures is only the linear probability weighted combination of losses beyond VaR. As we know, it is necessary to consider different orders of moments of a non-normal distribution in order to describe it comprehensively. Therefore, it should be better to consider the probability weighted combination of higher order losses below some critical value. In respect of risk measurement, the expectation of suitable order of lower tail losses is useful for describing the investor's degree of risk aversion, taking higher order losses is also helpful for controlling fat tails in the loss distribution (Chen and Wang, 2007). The only four papers, which mentioned the application of higher moments in the coherent risk measure construction, are Delbaen (2002), Fischer (2001, 2003) and Tasche (2002). It is briefly discussed in an example in Delbaen (2002) about the possible use of higher moments for defining coherent risk measures. Given in Fischer (2001, 2003) are some examples of coherent risk measures defined by directly combining the negative mean and weighted one-sided lower moments. It is also mentioned in Remark 3.8 in Tasche (2002) about the incorporation of higher moment effects in ES. In these theoretical oriented papers, the authors did not discuss the application of those risk measures to portfolio optimization.

In all the above mentioned LPM measures and coherent risk measures, the attention is put on either the demand side or the offer side. Only one-sided, that is, the lower part distribution information is used in the associated risk measure. Nevertheless, from the game theory point of view, the investor should take into account the specific action of the seller of a stock when he/she wants to buy that stock. This means that the attitudes towards potential losses from both demand and offer sides have to be taken into account simultaneously. In other words, to suitably measure investment risk and thereafter to make robust investment decision, a more general "two-sided" risk measure should be adopted. Meanwhile, as an unwanted side-effect of the strict separation between downside risk and upside potential, data are thrown away in the estimation process of LPM and quantile-based measures. This partial use of data renders those measures much more susceptible to estimation risk than their full domain analogue. Therefore, from the practical application point of view, it is also necessary for us to use some kind of two-sided or full domain risk measure. Preferably, such a measure will satisfy coherency conditions.

The only "two-sided" risk measure we have seen so far is that in Hamza and Janssen (1998), where the risk is measured through the convex combination of semivariances of the portfolio rate of return. Although the degree of risk aversion can be reflected by selecting the combination coefficient, the corresponding adjustment is only linear. This measure is still symmetrical in the sense that semivariances are used for measuring both below and above deviations from the mean. These two facts cause it difficult to represent the different roles of demand and offer sides and to control the fat-tail phenomenon. Last but not least, this risk measure is not a coherent measure.

Instead of measuring the overall seriousness of possible losses associated with random payoffs as that in coherent risk measures, general deviation measures are recently introduced in Rockafellar et al. (2006) to measure the pure uncertainty in the random payoff. This kind of measures need not be symmetric between "ups" and "downs". It is shown that lower range bounded deviation measures correspond one-to-one with coherent, strictly expectation bounded risk measures. Many interesting examples of deviation measures are derived from variants of CVaR and from basic error functionals in Rockafellar et al. (2006). Moreover, Rockafellar et al. (2007) lately demonstrated the existence of equilibrium in a financial market with investors using a diversity of deviation measures.

For real applications, what we mainly concern should be the practicality of the proposed risk measure for finding robust and efficient portfolios. Just recently, Quaranta and Zaffaroni (in press) applied the robust optimization technique to the minimization of the CVaR and the corresponding portfolio selection problem. Except for this, most existing theoretical papers about coherent risk measure (such as Acerbi, 2002; Fischer, 2003; Rockafellar et al., 2006) did not consider the application of the proposed risk measure for making optimal investment decision, needless to say the realistic portfolio selection problem with multiple market frictions taken into account simultaneously.

Bearing in mind the above limitations in current risk measures, a new class of two-sided coherent risk measures is constructed in this paper by properly combining the downside and the upside of the random payoff. As an application-oriented research and along a new derivation way, our new measure can be regarded as the improvement on preliminary coherent risk measures in Acerbi (2002), Fischer (2001, 2003) and as a variant of deviation measures in Rockafellar et al. (2006). Compared with existing risk measures, our new measure has the following advantages: the whole domain loss distribution information is utilized, which makes the new measure superior for finding robust (with respect to both trading sides) and stable (with respect to the estimation error) investment decisions; by suitably selecting the convex combination coefficient and the order of the norm of the downside random loss, it is easy for our new measure to reflect the investor's risk attitude and to control the asymmetry and fat tails of the loss distribution. Most importantly, it is easy to compute the new measure's value and to apply it to find the optimal portfolio. We will demonstrate these points by establishing a portfolio selection model with multiple market friction constraints and applying it to find robust optimal portfolios in real application.

2. The new risk measure and its properties

We consider a one-period framework, which means that we have the current date 0 and a future date D. No trading is possible between 0 and D. Therefore, "risk" here can be represented by the random payoff X (that is, the random profit if $X \geqslant 0$, or loss if $X < 0$), defined on a probability space $(\Omega, \mathscr{F}, P)$, of some assets or portfolios at D. Measuring risk is then equivalent to establishing a correspondence ρ between the space of X and the set of real numbers. To avoid tediously mathematical treatment, we assume that X is an p-integrable random variable. X can thus be treated as an element of the space $L^p(\Omega, \mathscr{F}, P)$ for $1 \leqslant p \leqslant \infty$.

In what follows, the risk measure $\rho(X)$ is considered to be the minimum extra cash added to X that makes the position acceptable for the holder (Artzner et al., 1999). In this sense, a positive $\rho(X)$ means that the investor has to add $\rho(X)$ amount of extra cash to ensure the acceptance of his future position, a negative $\rho(X)$ means that the investor can withdraw $-\rho(X)$ amount of money without affecting the acceptance of his future position. As usual, we define $\|X\|_p = (E_Q|X|^p)^{1/p}$, $\|X\|_\infty = ess.sup\{|X|\}$. Let X^- denote $\max(-X, 0)$, and X^+ denote $(-X)^-$. Lastly, define $\sigma_p^{\pm}(X) = \|(X - E_Q[X])^{\pm}\|_p$.

For reasons demonstrated in the last section, we propose here a new kind of two-sided risk measures that takes into account both sides of the loss distribution. Relative to the expected value $E_Q[X]$, the random variable $(X - E_Q[X])^-$ is the "downside" of X, which