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Master funds in portfolio analysis with general deviation measures

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Abstract

Generalized measures of deviation are considered as substitutes for standard deviation in a framework like that of classical portfolio theory for coping with the uncertainty inherent in achieving rates of return beyond the risk-free rate. Such measures, derived for example from conditional value-at-risk and its variants, can reflect the different attitudes of different classes of investors. They lead nonetheless to generalized one-fund theorems in which a more customized version of portfolio optimization is the aim, rather than the idea that a single "master fund" might arise from market equilibrium and serve the interests of all investors.

The results that are obtained cover discrete distributions along with continuous distributions. They are applicable therefore to portfolios involving derivatives, which create jumps in distribution functions at specific gain or loss values, well as to financial models involving finitely many scenarios. Furthermore, they deal rigorously with issues that come up at that level of generality, but have not received adequate attention, including possible lack of differentiability of the deviation expression with respect to the portfolio weights, and the potential nonuniqueness of optimal weights.

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The results also address in detail the phenomenon that if the risk-free rate lies above a certain threshold, the usually envisioned master fund must be replaced by one of alternative type, representing a "net short position" instead of a "net long position" in the risky instruments. For nonsymmetric deviation measures, the second type need not just be the reverse of the first type, and there can sometimes even be an interval for the risk-free rate in which no master fund of either type exists. A notion of basic fund, in place of master fund, is brought in to get around this difficulty and serve as a single guide to optimality regardless of such circumstances.

Keywords: Deviation measures; Risk measures; Value-at-risk; Conditional value-at-risk; Portfolio optimization; One-fund theorem; Master fund; Basic fund; Efficient frontier; Convex analysis

1. Introduction

In classical portfolio theory, investors respond to the uncertainty of profits by selecting portfolios that minimize variance, or equivalently standard deviation, subject to achieving a specified level in expected gain (Markowitz, 1952; Kroll et al., 1984; Markowitz, 1991). The well-known "one-fund theorem" (Tobin, 1958; Sharpe, 1964) stipulates that this can be accomplished in terms of a single "master fund" portfolio by means of a formula which balances the amount invested in that portfolio with the amount invested at the current risk-free rate. Nowadays, other approaches to uncertainty have gained in popularity. Portfolios are being selected on the basis of percentile characteristics such as *value-at-risk* (VaR), *conditional value-at-risk* (CVaR), or other properties proposed for use in risk assessment; cf. (Acerbi and Simonetti, 2002; Konno and Shirakawa, 1994; Malevergne and Sornette, 2002) and earlier alternatives such as in Bawa and Lindenberg (1977). These measures have no pretension to being universal, however; VaR and CVaR depend, for instance, on the specification of a confidence level parameter, which could vary among investors. Instead, what is apparent in the alternative approaches currently being touted is a move toward a kind of partial *customization* of responses to risk, while still avoiding, as impractical, a reliance on specifying individual utility functions.

A question in this evolving environment is the extent to which classical facts persist when the minimization of standard deviation is replaced by the minimization of some "nonstandard deviation." Researchers have already looked into the possibilities in special cases under various limiting assumptions (recognized explicitly or imbedded implicitly). Our goal, in contrast, is to demonstrate that important parallels with classical results exist much more broadly, despite technical hurdles, and in this way to bring out features that have not completely been analyzed, or even perceived, in the past.

We focus on the general *deviation measures* we developed axiomatically in Rockafellar et al. (2002a). Our idea is to substitute such a deviation measure for standard deviation in the setting of classical theory and investigate the consequences rigorously in detail. Furthermore, we aim at doing so, for the first time, in cases where