Economic Modelling 35 (2013) 401–408

Contents lists available at ScienceDirect

Economic Modelling

journal homepage: www.elsevier.com/locate/ecmod

Multi-period portfolio optimization under possibility measures

Xili Zhang [a], Weiguo Zhang [b], Weilin Xiao [a,*]

[a] School of Management, Zhejiang University, Hangzhou, China
[b] School of Business Administration, South China University of Technology, Guangzhou, China

ARTICLE INFO

Article history:
Accepted 16 July 2013

Keywords:
Multi-period portfolio selection
Possibility theory
Central value
Partial swarm optimization (PSO)

ABSTRACT

A single-period portfolio selection theory provides optimal tradeoff between the mean and the variance of the portfolio return for a future period. However, in a real investment process, the investment horizon is usually multi-period and the investor needs to rebalance his position from time to time. Hence it is natural to extend the single-period fuzzy portfolio selection to the multi-period case based on the possibility theory. In this paper, we propose the possibilistic expected value and variance for the terminal wealth with fuzzy forms after T periods by using the central value operator. Classes of multi-period possibilistic mean-variance models are formulated originally under the assumption that the proceeds of risky assets are fuzzy variables. Besides, we apply a particle swarm optimization algorithm to solve the proposed multi-period fuzzy portfolio selection models. A numerical example is given to illustrate the performance of the proposed models and algorithm.

1. Introduction

Portfolio selection is seeking the best allocation of wealth among different assets. Numerous studies on portfolio selection are based on the probabilistic mean-variance methodology first proposed by Markovitz (1952), such as, Perold (1984), Pang, (1980), Best (2010) and Maringer and Kellerer (2003). It has also gained widespread acceptance as a practical tool for portfolio optimization. However, it is a single period model which makes a one-off decision at the beginning of the period and holds on until the end of the period, while it was natural to extend Markowitz's work to multi-period portfolio selections, such as Smith (1967), Mossin (1968), Merton (1969), Samuelson (1969), Fama (1970), Hakansson (1971), Elton and Gruber (1974), Francis and Kirzner (1991), Dumas and Luciano (1991), Östermark (1991), Grauer and Hakansson (1993), Pliska (1997), Li and Ng (2000) and Chen (2005). The literatures mentioned often used the probability distribution of asset returns.

However, in the real world, the financial market behavior is affected by several non-probabilistic factors such as vagueness and ambiguity (see (Lacagnina and Pecorella, 2006)). The returns of assets are usually affected by many factors including economic, social, political and people's psychological factors as proposed by Huang (2011). Decision-makers are usually provided with information which is characterized by vague linguistic descriptions such as high risk, low profit, and high interest rate. In these cases, it is impossible for us to get the precise probability distribution we need. Furthermore, even if we know all the historical and current data, it is difficult that we predict the future return as a fixed value. Hence we need to consider that the future return has ambiguousness. There are several approaches dealing with ambiguous situations. On the one hand, some authors characterize uncertain distributions by defining a confidence region of their first two moments, so that the portfolio is robust against such uncertainty, see Pflug and Wozabal (2007) and Wozabal (2012).

On the other hand, fuzzy set theory and Possibility theory, proposed by Zadeh (1978) and advanced by Dubois et al. (1988), may help to solve problems in uncertain and imprecise environments. In particular, in the field of portfolio selection, investors are faced with forecasting the performance of the assets they manage. Given the uncertainty inherent in financial markets, analysts are very cautious in expressing their guesses. Hence, there exist a lot of published works in the field of finances, which incorporate the approach of fuzzy set theory.

Watada (1997) and León et al. (2002) discussed portfolio selection by using fuzzy decision theory. Inuiguchi and Tanino (2000) introduced a possibilistic programming approach to the portfolio selection problem under the minimax regret criterion. Carlsson et al. (2002) and Zhang et al. (2009a) introduced a possibilistic approach for selecting portfolios with the highest utility value under the assumption that the returns of assets are trapezoidal fuzzy numbers. Zhang et al. (2009b) discussed portfolio selection problem under possibilistic mean-variance utility and presented a SMO algorithm for finding the optimal solution. Zhang et al. (2010) proposed a risk tolerance model for portfolio adjusting problem with transaction costs based on possibilistic

* Corresponding author. Tel./fax: +86 571 88206867.
E-mail addresses: zhangxili831@gmail.com (X. Zhang), wgzhang@scut.edu.cn (W. Zhang), wlxiao81@gmail.com (W. Xiao).

0264-9993/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.econmod.2013.07.023

moments. Gupta et al. (2008) applied multi criteria decision making via fuzzy mathematical programming to develop comprehensive models of asset portfolio optimization for the investors pursuing either of the aggressive or conservative strategies. In addition, Bilbao-Terol et al. (2006), Zhang and Wang (2008), Li and Xu (2009), Huang (2010) and Zhang et al. (2011) discussed the portfolio selection problems in a fuzzy uncertain environment by following the ideas of probabilistic mean-variance model.

However, the researches mentioned above are on the single-period portfolio selection problems in fuzzy environment. There have been few literatures on multi-period fuzzy portfolio selection based on possibility theory. Zhang et al. (2012) presented a mean-semivariance-entropy model for multi-period portfolio selection based on possibility theory, in which the risk level is characterized by the sum of the lower possibilistic semivariance of portfolio return in each period. The aim of this paper is to develop a multi-period mean-variance portfolio selection model with fuzzy returns based on possibility theory. By using the central value operator introduced by Fullér and Majlender (2004) and Fullér et al. (2010a,b, 2011), we formulate the possibilistic expected value and possibilistic variance for the terminal wealth after T periods. A class of multi-period possibilistic mean-variance models is formulated originally. Moreover, an efficient solution is achieved for this class of fuzzy multi-period portfolio selection formulation, which makes the derived investment strategy an easy implementation task.

The organization of this paper is as follows. Section 2 introduces some basics of possibility distributions. In Section 3, we develop a class of multi-period portfolio selection models based on possibility theory. The optimization models are converted into crisp forms when the return of risky assets is taken as symmetrical triangular fuzzy variables. An efficient solution is derived in Section 4 to generate the optimal portfolio policy. In Section 5, an example is given to illustrate the behavior of the proposed models and algorithm. This paper concludes in Section 6 with some suggestions for future study.

2. Preliminaries

In this section, we will briefly recall some basics of possibility distributions which are crucial for our study (see (Carlsson and Fullér, 2011), for details). In general, the arithmetic operations on fuzzy numbers can be approached either by the direct use of the membership function (by the Zadeh extension principle) or by the equivalent use of the γ-cut representation (introduced by Goetschel et al. (1986)).

A fuzzy number A is a fuzzy set in $\mathbb{R}$ that has a normal, fuzzy convex and continuous membership function of bounded support $\mu_A:\mathbb{R}\rightarrow[0,1]$.The family of all fuzzy numbers will be denoted by $\mathcal{F}$. Fuzzy numbers can be considered as possibility distributions. If C is a fuzzy set in $\mathbb{R}^n$ then its γ-level set is defined by

$$[C]^\gamma=\begin{cases}\{x\in\mathbb{R}^n:\mu_C(x)\geq\gamma\}, & \gamma>0,\\ cl\{x\in\mathbb{R}^n:\mu_C(x)\geq\gamma\}, & \gamma=0,\end{cases}$$

where $cl(C)$ means the closure of support of C. It is clear that if $A\in\mathcal{F}$ is a fuzzy number then $[A]^\gamma$ is a convex and compact subset of $\mathbb{R}$ for all $\gamma\in[0,1]$, i.e. $[A]^\gamma=[A_L(\gamma),A_U(\gamma)]$, where $A_L(\gamma)=\inf\{x\in\mathbb{R}:\mu_A(x)\geq\gamma\}$ and $A_U(\gamma)=\sup\{x\in\mathbb{R}:\mu_A(x)\geq\gamma\}$.

Let $A_i\in\mathcal{F}, i=1,\ldots,n$ be fuzzy numbers, and let C in $\mathbb{R}^n$ be a fuzzy set. As discussed in Carlsson et al. (2005) and Fullér and Majlender (2004), fuzzy set C is said to be a *joint possibility distribution* of fuzzy numbers A_i, $i=1,\ldots,n$, if it satisfies the relationship

$$\sup_{x_j\in\mathbb{R},j\neq i}\mu_C(x_1,\ldots,x_n)=\mu_{A_i}(x_i),\quad\forall x_i\in\mathbb{R},i=1,\ldots,n.$$

Furthermore, A_i is called the *ith marginal possibility distribution* of C and notated $A_i=\pi_i(C)$, where π_i denotes the projection operator in $\mathbb{R}^n$ on the ith axis, $i=1,\ldots,n$. That is the marginal possibility distributions are derived by the projection principle from a joint possibility distribution; the projection of C on the ith axis is A_i for $i=1,\ldots,n$.

In the sense of subsethood of fuzzy sets the largest joint possibility distribution defines the concept of *non-interaction*. Fuzzy numbers $A_i\in\mathcal{F}, i=1,\ldots,n$ are said to be non-interactive if their joint possibility distribution is given by

$$\mu_C(x_1,\ldots,x_n)=\min_i\left\{\mu_{A_i}(x_i)\right\},\forall x_1,\ldots,x_n\in\mathbb{R},$$

and the equality $[C]^\gamma=[A_1]^\gamma\times\ldots\times[A_n]^\gamma$ holds for all $\gamma\in[0,1]$.

In the following, we present the definition of the central value which is given in Fullér and Majlender (2004).

Definition 2.1. Let C be a joint possibility distribution in $\mathbb{R}^n$, let $g:\mathbb{R}^n\rightarrow\mathbb{R}$ be an integrable function, and let $\gamma\in[0,1]$. Then, the central value of g on $[C]^\gamma$ is defined by

$$\begin{aligned}\Psi_{[C]^\gamma}(g)&=\frac{1}{\int_{[C]^\gamma}dx}\int_{[C]^\gamma}g(x)dx\\&=\frac{1}{\int_{[C]^\gamma}dx_1\ldots dx_n}\int_{[C]^\gamma}g(x_1,\ldots,x_n)dx_1\ldots dx_n,\end{aligned}$$

for all $\gamma\in[0,1]$, where Ψ is called as the central value operator.

It is obvious that for any fixed possibility distribution C and $\gamma\in[0,1]$ $\Psi_{[C]^\gamma}$ is a linear operator. Especially, if $n=1$ and $g=x$ is the identity function ($g=id$) over $\mathbb{R}$, then for any fuzzy number $A\in\mathcal{F}$ with $[A]^\gamma=[a_1(\gamma),b_1(\gamma)],\gamma\in[0,1]$, the central value of the identity function is computed by

$$\Psi_{[A]^\gamma}(id)\equiv\Psi\left([A]^\gamma\right)=\frac{1}{\int_{[A]^\gamma}dx}\int_{[A]^\gamma}xdx=\frac{a_1(\gamma)+b_1(\gamma)}{2}. \tag{2.1}$$

Let us denote the projection functions on $\mathbb{R}^2$ by π_x and π_y, i.e., $\pi_x(u,v)=u$ and $\pi_y(u,v)=v$ for all $u,v\in\mathbb{R}$. In what follows, we will show two important properties of the central value operator.

Lemma 2.1. *If $A,B\in\mathcal{F}$ are non-interactive and $g=\pi_x+\pi_y$ is the addition operator on $\mathbb{R}^2$ then*

$$\Psi_{[C]^\gamma}\left(\pi_x+\pi_y\right)=\Psi_{[A]^\gamma}(id)+\Psi_{[B]^\gamma}(id)=\Psi\left([A]^\gamma\right)+\Psi\left([B]^\gamma\right),$$

for all $\gamma\in[0,1]$.

Lemma 2.2. *If $A,B\in\mathcal{F}$ are non-interactive and $g=\pi_x\pi_y$ is the multiplication operator on $\mathbb{R}^2$ then*

$$\Psi_{[C]^\gamma}\left(\pi_x\pi_y\right)=\Psi_{[A]^\gamma}(id)\Psi_{[B]^\gamma}(id)=\Psi\left([A]^\gamma\right)\Psi\left([B]^\gamma\right),$$

for all $\gamma\in[0,1]$.

Based on Definition 2.1, the measure of dispersion is defined in Carlsson et al. (2005) as follows.

Definition 2.2. Let A be a possibility distribution in $\mathbb{R}$, and let $\gamma\in[0,1]$. Then the measure of *dispersion* of $[A]^\gamma$ is defined by

$$\mathcal{R}_{[A]^\gamma}(id,id)=\Psi_{[A]^\gamma}\left(id-\Psi_{[A]^\gamma}(id)\right)^2=\Psi_{[A]^\gamma}\left(id^2\right)-\Psi^2_{[A]^\gamma}(id),$$

for all $\gamma\in[0,1]$.