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Optimal retirement consumption with a stochastic force of mortality[☆]

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ABSTRACT

We extend the lifecycle model (LCM) of consumption over a random horizon (also known as the Yaari model) to a world in which (i) the force of mortality obeys a diffusion process as opposed to being deterministic, and (ii) consumers can adapt their consumption strategy to new information about their mortality rate (also known as health status) as it becomes available. In particular, we derive the optimal consumption rate and focus on the impact of mortality *rate* uncertainty versus simple *lifetime* uncertainty — assuming that the actuarial survival curves are initially identical — in the retirement phase where this risk plays a greater role.

In addition to deriving and numerically solving the partial differential equation (PDE) for the optimal consumption rate, our main general result is that when the utility preferences are logarithmic the initial consumption rates are identical. But, in a constant relative risk aversion (CRRA) framework in which the coefficient of relative risk aversion is greater (smaller) than one, the consumption rate is higher (lower) and a stochastic force of mortality does make a difference.

That said, numerical experiments indicate that, even for non-logarithmic preferences, the stochastic mortality effect is relatively minor from the individual's perspective. Our results should be relevant to researchers interested in calibrating the lifecycle model as well as those who provide normative guidance (also known as financial advice) to retirees.

1. Introduction and motivation

The lifecycle model (LCM) of savings and consumption – originally postulated by Fisher (1930) and refined by Modigliani and Brumberg (1954) and Modigliani (1986) – is at the core of most multi-period asset pricing and allocation models, as well as being the foundation of microeconomic consumer behavior. The original formulation – see, for example, Ramsey (1928) and Phelps (1962) – assumed a deterministic horizon. But, in a seminal contribution, the LCM was extended by Yaari (1964, 1965) to a stochastic lifetime, which eventually led to the models of Merton (1971), Richard (1975) and hundreds of subsequent papers on asset allocation over the human lifecycle.

The conceptual underpinning of the LCM is the intuitive notion of *consumption smoothing*, whereby (rational) individuals seek to minimize disruptions to their standard of living over their entire life. They plan a consumption profile that is continuous, equating marginal utility at all points, based on the assumption of a concave utility function. See the recent (and very accessible) article by Kotlikoff (2008) in which this concept is explained in a non-technical way.

Once again, until the seminal contribution by Yaari (1964, 1965), the LCM was employed by economists in an idealized world in which death occurred with probability one at some terminal horizon. Menahem Yaari introduced lifetime uncertainty into the lifecycle model, in addition to his more widely known contribution of introducing actuarial notes and annuities into optimal consumption theory.

In the expressions (and theorems) he derived for the optimal consumption function, Yaari (1965) assumed a very general *force of mortality* for the remaining lifetime random variable, without specifying a particular law. His results would obviously include a constant force of mortality (i.e. exponential remaining lifetime) as well as Gompertz–Makeham (GM) mortality, and other commonly formulated approximations. Yaari provided a rigorous foundation for Irving Fisher's claim that lifetime uncertainty effectively increases consumption impatience and is akin to behavior under

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higher subjective discount rates. Mathematically, the mortality rate was added to the subjective discount rate.

That said, most of the empirical or prescriptive papers in the LCM literature have not gone beyond assuming the GM law – or some related deterministic function – for calibration purposes. In other words, mortality is just a substitute for subjective discount rates. In fact, one is hard pressed to differentiate high levels of longevity risk aversion from weak preferences for consumption today versus the future. Some have labeled this risk neutrality with respect to lifetime uncertainty.

For example, Levhari and Mirman (1977), Davies (1981), Deaton (1991), Leung (1994), Butler (2001), Bodie et al. (2004), Dybvig and Liu (2005), Kingston and Thorp (2005), Babbel and Merrill (2006), Park (2006), Wallmeier and Zainhofer (2007), Feigenbaum (2008), and the recent work by Lachance (2012), all assume a deterministic force of mortality.

Indeed, some economists continue (surprisingly) to ignore mortality altogether, for example the recent review by Attanasio and Weber (2010). Perhaps this is because, when the force of mortality is deterministic, it can be added to the subjective discount rate without any impact on the mathematical structure of the problem.

To our knowledge, the only authors within the financial economics literature that have considered the possibility of non-constant mortality rates in a lifecycle model are Cocco and Gomes (2009), although their Lee–Carter mortality model is not quite *stochastic* as in Milevsky and Promislow (2001), Dahl (2004), Cairns et al. (2006), or the various models described in the book by Pitacco et al. (2008), or the concerns expressed by Norberg (2010).

Moreover, a number of very recent papers – for example Menoncin (2008), Stevens (2009) and Post (2010) – have examined the implications of (truly) stochastic mortality rates on the demand and pricing of certainly annuity products, but have not derived the impact of stochasticity on optimal consumption alone or examined the impact of pure uncertainty in the mortality rate.

Another related paper is that of Bommier and Villeneuve (2012), who examine the impact of relaxing the assumption of additively separable utility and what they call risk neutrality with respect to life duration. But, they also assume a deterministic force of mortality in their formulation and examples. In that sense, our work is similar because we also relax the so-called risk neutrality and the intertemporal additivity.

In sum, to our knowledge, none of the existing papers within the LCM literature have assumed a stochastic force of mortality – which is the model of choice in the current actuarial and insurance literature – and then derived its impact on pure consumption behavior. We believe this to be a foundational question, and in this paper our objective is straightforward, namely, to compare the impact of stochastic versus deterministic mortality rates on the optimal consumption rate.

1.1. A proper comparison

Assume that two hypothetical retirees – i.e. consumers who are not expecting any future labor income – approach a financial economist for guidance on how they should spend their accumulated financial capital over their remaining lifetime, a time horizon they both acknowledge is stochastic. Assume that both retirees have time-separable and rational preferences and seek to maximize discounted utility of lifetime consumption under the same elasticity of intertemporal substitution ($1/\gamma$), the same subjective discount rate (ρ), and the same initial financial capital constraint (F_0). They have no declared bequest motives and – for whatever reason – neither is willing (or able) to invest in anything other than a risk-free asset with instantaneous return (r); which means they are *not* looking for guidance on asset allocation or annuities.[1] All they want is an optimal consumption plan ($c^*(t)$; $t \geq 0$) guiding them from time zero (retirement) to the last possible time date of death ($t \leq D$). Most importantly, both retirees agree that they share the same probability-of-survival curve, denoted by $p(s)$. In other words, they currently live in the same health state, and have the same effective biological age. For example, they both agree on a $p(35) = 5\%$ probability that either of them survives for 35 years and a $p(20) = 50\%$ probability that either of them survives for 20 years, etc.

Yaari (1964, 1965) showed exactly how to solve such a problem. He derived the Euler–Lagrange equation for the optimal trajectory of wealth and the related consumption function.

In Yaari's model, both of the above-mentioned retirees would be told to follow identical consumption paths until their random date of death. In fact, they would both be guided to optimally consume $c(t)^* = F(t)/a(t)$, where $a(t)$ is a *function of time only* and is related to an actuarial annuity factor. We will explain this factor in more detail later in the paper.

But here is the impetus for our comparison. Although both retirees appear to have the same longevity risk assessment and agree on the survival probability curve $p(s)$, they have *differing views about the volatility of their health as proxied by a mortality rate volatility*. In the language of current actuarial science, the first retiree (1) believes that his/her instantaneous force of mortality (denoted by $\lambda^{\mathrm{DfM}}(t)$) will grow at a deterministic rate until he/she eventually dies, while the second retiree (2) believes that his/her force of mortality (denoted by $\lambda^{\mathrm{SfM}}(t)$) will grow at stochastic (but measurable) rate until a random date of death. As such, the remaining lifetime random variable for retiree 2 is doubly stochastic. While this distinction might sound farfetched and artificial, a growing number of researchers in the actuarial literature are moving to such models,[2] rather than the simplistic mortality models traditionally used by economists. The actuaries' motivation in advocating for a stochastic force of mortality (SfM) is to generate more robust pricing and reserving for mortality-contingent claims. These studies have all argued that SfM models better reflect the uncertainty inherent in demographic projections *vis à vis* the inability of insurance companies to diversify mortality risk entirely. We ask: *how do the recent actuarial models impact the individual economics of the problem?*

When one thinks about it, real-life mortality rates are indeed stochastic, capturing (unexpected) improvements in medical treatment, or (unexpected) epidemics, or even (unexpected) changes to the health status of an individual. Rational consumers choosing to make saving and consumption decisions using models based on deterministic mortality rates would likely agree to reevaluate those decisions if their views about the values of those mortality rates change dramatically. Our thesis is that economic decision making can only be improved if mortality models reflect the realistic evolution of mortality rates.

We will carefully explain the mathematical distinction between deterministic and stochastic forces of mortality (DfMs and SfMs) in Section 2 of this paper, but just to make it clear here, at time zero both our hypothetical retirees agree on the initial survival probability curve $p(s)$. However, at any future time their perceived survival probability curves will deviate from each other depending on the realization of the mortality rate between now and then.

Motivated by such models of mortality, in this paper we derive the optimal consumption function for both retirees; one

1 This simplification is made purely to focus attention on the impact of stochastic mortality.

2 We appreciate and acknowledge comments made by a referee, that models in which mortality depends on health status, which itself is stochastic, were used by actuaries well before the introduction of 21st century stochastic mortality models.