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An optimal investment, consumption, leisure, and voluntary retirement problem with Cobb–Douglas utility: Dynamic programming approaches

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ABSTRACT

We consider an optimal consumption, leisure, investment, and voluntary retirement problem for an agent with a Cobb–Douglas utility function. Using dynamic programming, we derive closed form solutions for the value function and optimal strategies for consumption, leisure, investment, and retirement.

1. Introduction

We consider an optimal consumption, leisure, and investment problem with voluntary retirement for an agent whose period utility function is a Cobb–Douglas utility function of consumption and leisure. In this model the agent can flexibly choose her leisure amount before retirement above a certain minimum labor requirement, and will receive labor income proportional to the amount of labor supplied. Upon retirement, the agent will enjoy full leisure, at the cost of forgoing all labor income. Using the dynamic programming method pioneered by Merton [1,2] and Karatzas et al. [3] we find closed form solutions to the value function and find the optimal consumption, leisure, and portfolio policies.

Barucci and Marazzina [4] consider this consumption, leisure, investment, and retirement problem in the case of stochastic labor income. Choi et al. [5] solve a similar problem for an agent who has constant elasticity of substitution (CES) period utility. Farhi and Panageas [6] also consider such a problem, where the choice of leisure is confined to only two values: l_1 while working and $\bar{l}$ after retirement. In all of these papers, the authors use the martingale method to solve their optimization problems (see also [7,8]). Shin [9] extends the results of Farhi and Panageas [6] by solving their problem using the dynamic programming method, and shows the equivalence of the solutions obtained through the martingale method and the dynamic programming method. Likewise, we provide a methodological contribution by solving our optimization problem using the dynamic programming method.

The work is organized as follows. Section 2 provides information on the financial market. Section 3 lays out and solves our optimization problem, with detailed proofs provided.

2. The financial market

In our continuous time financial market, we assume that there are two assets: a riskless asset $S_0(\cdot)$ and a risky asset $S_1(\cdot)$, which follow $dS_0(t)/S_0(t) = rdt$ and $dS_1(t)/S_1(t) = \mu dt + \sigma dB(t)$, respectively. The parameters $r > 0$,

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$\mu > r$ and σ are assumed to be constants, and $B(\cdot)$ is a standard Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The market price of risk is defined by $\theta := (\mu - r)/\sigma$. Let π_t be the dollar-amount invested in the risky asset $S_1(t)$ at time t, c_t be the consumption rate process at time t, l_t be the leisure rate process at time t, and τ be a $\mathcal{F}_t$-stopping time considered as a voluntary retirement time from labor. π_t, c_t, and l_t are $\mathcal{F}_t$-progressively measurable with $\int_0^t \pi_s^2 ds < \infty$, $\int_0^t c_s ds < \infty$, and $\int_0^t l_s ds < \infty$, for all $t \geq 0$ almost surely.

For before retirement, we consider l_t to be a control variable under the restriction $l_t \leq L < \bar{L}$. For after retirement, however, $l_t = \bar{L}$ is considered to be a constant. Let w be a constant wage rate, and let us define $w(\bar{L} - l_t)$ as labor income at time t. Thus the wealth process is given by

$$dX_t = \left[rX_t + \pi_t(\mu - r) - c_t + w(\bar{L} - l_t)\right]dt + \sigma\pi_t dB(t), \qquad X_0 = x > -w\bar{L}/r,$$

where $w\bar{L}/r$ denotes the present value of the future labor income of the agent. The agent can consume and invest as long as $x > -w\bar{L}/r$.

3. The optimization problem

The agent in our model wants to maximize her expected utility

$$V(x) = \max_{(c,l,\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}u(c_t, l_t)dt\right] = \max_{(c,l,\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}u(c_t, l_t)dt + e^{-\rho\tau}U(X_\tau)\right], \tag{3.1}$$

where $\rho > 0$ is a subjective discount rate and $\mathcal{A}(x)$ is an admissible plan of quadruples (c, l, π, τ) such that

$$\mathbb{E}\left[\int_0^\tau e^{-\rho t}u^-(c_t, l_t)dt + e^{-\rho\tau}U^-(X_\tau)\right] < \infty,$$

where $u^- := \max(-u, 0)$. $u(c, l)$ is a Cobb–Douglas utility function defined by

$$u(c, l) := \frac{1}{\alpha} \cdot \frac{\left(c^\alpha l^{1-\alpha}\right)^{1-\gamma}}{1-\gamma}, \quad 0 < \alpha < 1, \gamma > 0 \text{ and } \gamma \neq 1, \tag{3.2}$$

where γ is the agent's coefficient of relative risk aversion for two different goods c and l, and α is a constant parameter that measures the share of consumption's contribution to the agent's period utility. If we define $\gamma_1 := 1 - \alpha(1 - \gamma)$, then the Cobb–Douglas utility function (3.2) can be rewritten as

$$u(c, l) = \frac{c^{1-\gamma_1} l^{\gamma_1-\gamma}}{1-\gamma_1}.$$

The following assumption is needed to make our optimization problem well-defined and holds throughout the work without further comment.

Assumption 3.1.

$$K := r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 > 0 \quad \text{and} \quad K_1 := r + \frac{\rho - r}{\gamma_1} + \frac{\gamma_1 - 1}{2\gamma_1^2}\theta^2 > 0.$$

Remark 3.1. The post-retirement value function $U(\cdot)$ in (3.1) is similar to that of the classical Merton's problem, and can be obtained as follows:

$$U(X_\tau) = \frac{\bar{L}^{\gamma_1-\gamma}}{K_1^{\gamma_1}(1-\gamma_1)} X_\tau^{1-\gamma_1}.$$

Remark 3.2. For later consideration, we introduce a quadratic equation

$$f(m) := \frac{1}{2}\theta^2 m^2 + \left(\rho - r + \frac{1}{2}\theta^2\right)m - r = 0, \tag{3.3}$$

with two roots $m_+ > 0$ and $m_- < -1$.

The next theorem gives our main results.