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Optimal portfolio and background risk: an exact and an approximated solution

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Abstract

This paper analyses the portfolio problem of an investor maximizing the expected exponential utility of his terminal real wealth. The investor must cope with both a set of stochastic investment opportunities and a set of background risks. If the market is complete we are able to find an exact solution. If the market is incomplete, we suggest an approximated general solution. Contrary to other exact solutions obtained in the literature, all our results are obtained considering a stochastic inflation risk and without specifying any particular functional form for the stochastic variables involved in the problem.

JEL classification: G11; C61

Keywords: Asset allocation; Background risk; Inflation risk; Feynman–Kač theorem; Stochastic investment opportunities

1. Introduction

This work analyses the issue of optimal portfolio policy in a multi-period model where investors maximize the expected utility of their terminal wealth facing, in particular, some risks outside the financial market that we will refer to as "background risks". Furthermore, our work offers a contribution to the investment problem in the rather general case where the value of assets depends on the stochastic behaviour of a set of state variables.

The vector of state variables contains all the stochastic variables directly affecting the asset prices but indirectly affecting the investors' wealth. For a review of all variables which can affect the asset prices readers are referred to Campbell (2000) who offers a survey of the most important contributions in this field.

Besides these state variables, we consider a set of stochastic processes, called in the paper "background variables". They affect the level of investor's wealth and by means of these variables we are able to take into account a lot of particular cases. For instance, the vector of the background variables can contain (i) the investor's wage process affecting the investment strategies (see for instance Franke et al., 2001), (ii) the contributions (or the withdrawals) of the subscribers to a pension fund (see for instance Blake, 1998; Blake et al., 2000; Boulier et al., 2001), and (iii) the indemnities paid by an insurance company to the policy holders (see Young and Zariphopoulou, 2000).

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Furthermore, we also take into account the inflation risk. Actually, when a long period of time is considered, this risk cannot be neglected. Since all the above mentioned literature considers the case of an institutional investor, it seems quite important to take into account the problem of maximizing the real wealth instead of the nominal wealth. Accordingly, in our model, we propose to maximize the expected utility of the investor's terminal real wealth.

A lot of attempts have been made in the literature for introducing the background risk in the optimal portfolio problem (see the works above mentioned) but often the authors make a lot of strong assumptions in order to reach a framework which can be handled. In this work, the presence of the inflation risk can solve some of the problems which are generally faced in this type of models. In particular, we are able to solve some technicalities which make the problem very difficult to solve when the usual nominal approach is considered.

In this paper, we follow the traditional route to use the stochastic dynamic programming technique (Merton, 1969, 1971) leading to the Hamilton–Jacobi–Bellman (HJB) equation.[1] For the method called "martingale approach" the reader is referred to Cox and Huang (1989, 1991) and Lioui and Poncet (2001). We just underline that in this work we are able to reach the same qualitative results as Lioui and Poncet even if they do not consider any background risk.

In our framework, we find that the optimal portfolio is formed by three components: (i) a preference-free component minimizing the instantaneous variance of the investor's wealth differential and immunizing the investor's portfolio against the background risks, (ii) a speculative part proportional to both the Sharpe ratio of investor's portfolio and the inverse of the Arrow–Pratt risk aversion index, and (iii) a component depending on the derivatives of the value function (indirect utility function) with respect to all the state and background variables. The last component is the only one depending on the investor's time horizon.

In order to find the explicit solution for the value function, it is necessary to solve the HJB equation. Unfortunately, solving this highly non-linear partial differential equation is the most difficult task of the stochastic optimal control approach. In fact, some algebraic solutions can only be obtained in very special cases. In particular, we refer to the works of Kim and Omberg (1996), Wachter (1998), Boulier et al. (2001) and Deelstra et al. (2000).

In the present work we show that if the financial market is complete, then we can find an exact solution to our optimal portfolio problem with background risk. This result can be obtained thanks to the insertion of the inflation risk affecting the growth rate of investor's wealth. In fact, if this kind of risk exists, then the riskless asset is no more "riskless". In particular, if the investor wants to maximize the expected value of his terminal real wealth, then for him no riskless asset exists because the nominal riskless asset does not cover the portfolio against the inflation risk. Furthermore, we outline that, contrary to the exact solutions above mentioned, we do not specify any particular functional form for the behaviour of the stochastic variables involved in the problem and we reach, in this way, a very general solution indeed.

If the completeness hypothesis does not hold, then we propose a general approximated solution to the HJB equation. In particular, our work concentrates on the hypothesis that the value function has a suitable form under which the Feynman–Kač theorem can be applied to the HJB equation. Thus, even if in an incomplete market our solution is exact only under particular conditions that must hold on the value function, we find that it stays valid as an approximated solution under conditions which are not very restrictive.

We underline that the exact solutions presented in Kim and Omberg (1996), Boulier et al. (2001) and Deelstra et al. (2000), consider only one state variable and do not take into account any background risk. Instead, our model is able to determine both an exact and an approximated solution when there exists a set of generic state and background variables. Thus, our framework seems to be very general and able to be applied to many particular cases.

Through this work we consider agents trading continuously in a frictionless, arbitrage-free market until time H, which is the horizon of the economy. Furthermore, we analyse both a complete and an incomplete market structure.

[1] Øksendal (2000) and Björk (1998) offer a complete derivation of the HJB equation.