Available online at www.sciencedirect.com

Insurance: Mathematics and Economics 32 (2003) 331–344

INSURANCE MATHEMATICS & ECONOMICS

www.elsevier.com/locate/econbase

Risk comparisons of premium rules: optimality and a life insurance study

Søren Asmussen[a,*], Jakob R. Møller[b]

[a] *Mathematical Statistics, Center for Mathematical Sciences, Lund University, S-221 00 Lund, Sweden*
[b] *Højager 144, DK-7300 Jelling, Denmark*

Received June 2001; received in revised form November 2002; accepted 27 November 2002

Abstract

Consider a risk $Y_1(x)$ depending on an observable covariate x which is the outcome of a random variable A with a known distribution, and consider a premium $p(x)$ of the form $p(x) = \mathbb{E}Y_1(x) + \eta p_1(x)$. The corresponding adjustment coefficient γ is the solution of $\mathbb{E}\exp\{\gamma[Y_1(A) - p(A)]\} = 1$, and we characterize the rule for the loading premium $p_1(\cdot)$ which maximizes γ subject to the constraint $\mathbb{E}p_1(A) = 1$.

In a life insurance study, the optimal $p_1^*(\cdot)$ is compared to other premium principles like the expected value, the variance and the standard deviation principles as well as the practically important rules based on safe mortality rates (i.e., using the first order basis rather than the third order one). The life insurance model incorporates premium reserves, discounting, and interest return on the premium reserve but not on the free reserve. Bonus is not included either.

Keywords: Adjustment coefficient; Convex ordering; Delayed claims; First order basis; Gompertz–Makeham law; Large deviations; Life annuities; Loading premium; Shot noise; Third order basis; Whole life insurance

1. Introduction

A main topic in expositions of life insurance mathematics (e.g., Gerber, 1997; Bowers et al., 1997) is the calculation of equivalence premiums: given an insurance treaty such as whole life (with single or gradual premium) or life annuities, one is concerned with the identities coming from equating the (discounted) stream of payments from the insured to the company to the stream going the opposite way.

Once the equivalence premium is calculated, one must of course add a loading premium. However, this topic is only peripherically touched upon in the mathematical literature on life insurance. For example, all that we could find in Gerber (1997, p. 52) was the statement that 'Net premiums are nevertheless of utmost importance in insurance practice. Moreover, they are usually calculated on conservative assumptions about future interest and mortality, thus creating an implicit safety loading'. (Here 'conservative' should be interpreted as in favor of the company, say overestimating the mortality rates in the case of whole life insurance with a single premium and underestimating

* Corresponding author. Present address: Aarhus University, Aarhus, Denmark.
E-mail addresses: asmus@maths.lth.se (S. Asmussen), jrm@pc.dk (J.R. Møller).
URL: http://www.maths.lth.se/matstat/staff/asmus.

0167-6687/$ – see front matter © 2003 Published by Elsevier Science B.V.
doi:10.1016/S0167-6687(02)00208-1

them in the case of life annuities; see, e.g., Deis et al. (1993) for a discussion of the practical implementation by the Danish life insurance companies.)

This rudimentary treatment of the loading premium contrasts the literature on non-life insurance mathematics. Here any text (e.g., Sundt, 1993) inevitably introduces premium principles based upon say expected value, variance, standard deviation or utility, and go into a discussion of their merits from the point of view of both the insurer and the insured.

The present paper has as its aims to formalize a criterion for comparing premium rules in general insurance and go in more depth with the comparison in the special case of life insurance. The comparison is in terms of risk, more precisely the adjustment coefficient; this is a traditional choice in non-life insurance, but in life insurance it is not obvious that an adjustment coefficient exists and we will return to this point later on. We will consider risks $Y_1(x)$ depending on a background covariate x. In the portfolio, x is the outcome of an r.v. A (for any individual risk, x is observable, as opposed to say the standard setting of credibility theory). In life insurance, x is typically the age of the insured when the contract is signed, whereas in say fire insurance, x could be the (floor) space of the buildings insured. The equivalence premium is $p_0(x) = \mathbb{E}Y_1(x)$ and we write $Y_0(x) = Y_1(x) - p_0(x)$. We write the total premium charged as $p(x) = p_0(x) + \eta p_1(x)$ where η is a fixed loading constant and $p_1(\cdot)$ some arbitrary function. In order to be able to compare different premium rules, we assume a fixed loading:

$$\mathbb{E}p_1(A) = 1 \tag{1.1}$$

(the safety loading is then $\delta = \eta/\mathbb{E}p_0(A)$). For example, $p_1(x)$ is given by

$$\frac{p_0(x)}{\mathbb{E}p_0(A)}, \qquad \frac{\sigma^2(x)}{\mathbb{E}\sigma^2(A)}, \qquad \frac{\sigma(x)}{\mathbb{E}\sigma(A)}$$

for the expected value principle, the variance principle and the standard deviation principle, respectively, where $\sigma^2(x) = \mathbf{Var}(Y_1(x)) = \mathbf{Var}(Y_0(x))$. This list is not exhaustive; further principles are, e.g., the percentile principle and the utility principle which we do not discuss here.

Write $Y(x) = Y_1(x) - p(x) = Y_0(x) - \eta p_1(x)$ and $Y = Y^{(p)} = Y(A)$ where A is independent of $Y(x)$. Then Y is the total surplus of a typical insurance policy (note that the sign is chosen such that $Y > 0$ means a loss for the company and $Y < 0$ a gain).

The problem is to choose $p(\cdot)$ or, equivalently, $p_1(\cdot)$ so as to minimize the risk of the company subject to the constraint (1.1). Here *risk minimization* needs to be defined in some appropriate sense, and we consider two ways, variance minimization and adjustment coefficient maximization. We discuss this in more detail in Section 2; for the moment, it will suffice to be aware that the objective of adjustment coefficient maximization is to choose $p(\cdot)$ to maximize the solution $\gamma = \gamma(p)$ of

$$\mathbb{E}\,\mathrm{e}^{\gamma Y^{(p)}} = 1. \tag{1.2}$$

One first motivation for this comes from a discrete time random walk model:

$$R_n = u - V_1 - \cdots - V_n \tag{1.3}$$

for the reserve where u is the initial reserve and the V_k are i.i.d., such that V_i conditional upon $A_i = x$ has the same distribution as $Y(x)$, where $A_1, A_2, \ldots$ are i.i.d. distributed as A. Under the appropriate conditions on existence of exponential moments (which are tacitly assumed throughout the paper), the ruin probability $\psi(u)$ for this model is indeed asymptotically of the form

$$\psi(u) \sim C\,\mathrm{e}^{-\gamma u}, \tag{1.4}$$

where γ is the solution of (1.2). This model is certainly a very crude approximation since it ignores the complicated issue of delayed claims settlements as in life insurance. However, we will see later that a similar exponential