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# Moments of the cash value of future payment streams arising from life insurance contracts

Joanna Dębicka

*Department of Statistics and Economic Cybernetics, Wroclaw University of Economics, ul. Komandorska 118-120, 53-345 Wroclaw, Poland*

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**Abstract**

A model for the cash value of a stream of future payments arising from an insurance contract, where the interest rate and future-lifetime are random, is studied. A matrix form for formulas for the first two moments of the cash value of the stream of future payments for a portfolio of policies is derived. Numerical illustrations for the rate of interest modeled by a Wiener and an Ornstein–Uhlenbeck process are provided.


*PACS:* IM30; IB11; IB12

*Keywords:* Portfolio of policies; Cash value of the payment stream; Joint randomness in interest and mortality; Wiener process; Ornstein–Uhlenbeck process

## 1. Introduction

A typical insurance on the life of one person can be built up as a combination of two different forms of insurance contracts:

- life annuities paying fixed amounts at specific dates (provided that the annuitant is alive),
- life insurances paying a fixed amount at the death of the insurant.

In the actuarial theory, the future benefits (amounts paid according to annuity or life insurance) are discounted to the present (to time 0) by some interest rate $Y(t)$. This produces the *present value* of benefits, which is a random variable. In practice, it is important to find the mean of the present value of benefits with regard to the future-lifetime of the insured person, called the *actuarial value* of benefits. Large attention has been put to calculate the first two moments of the present value of benefits; since they play an important role for the valuation of premiums and determination of reserves of a portfolio of life insurance contracts. There is a vast literature which deals with the present value of benefits in a stochastic mortality and interest environment. In particular, for annuity contracts, Beekman and Fuelling (1990) obtained the expression for the mean value and standard deviation of the present value of future payment streams (for a continuous-time model) for some particular life annuity contracts. For a discrete-time model,

*E-mail address:* jdebicka@manager.ae.wroc.pl (J. Dębicka).


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the expected value and standard deviation of the present value of annuity-immediate are given in Parker (1994d). In the life insurance case, portfolio of temporary insurance contracts was analyzed by Parker (1994c), where moments of the present value of benefits are given. An extension of this model to a homogeneous portfolio of endowment insurance policies is presented in Parker (1994a).

On the other hand, life insurance may also be considered from the point of view of financial mathematics. In this setting the insurance policy gives rise to two payment streams. First, a stream of premium payments, which flows from the insured to the insurer. Second (in the opposite direction), a stream of actuarial payment functions, where fixed amounts under the life annuity product and fixed life insurance benefits are considered as a series of deterministic future cash flows. Thus the *cash value* of future payment stream, defined as the future cash flows arising from insurance contract discounted to time 0, is the analogon of the present value of benefit. For the analysis of moments of the cash value of future payment streams we refer to e.g. Hoem and Aalen (1978), Norberg (1993), Parker (1993). An axiomatic approach to interest and valuation of random payment streams arising from insurance contract is given in Norberg (1990). Expressions for the moments of the present value of the future cash flows arising from the benefit obligations of a general portfolio of life insurance policies for the discrete-time model (comprised temporary, endowment, pure endowment) are analyzed by Parker (1997).

It appears that formulas for the moments of the cash value of future payment streams are quite extensive, especially for the discrete-time case. The aim of this paper is to express the first two moments of the cash value of the future payment streams arising from insurance contract in a matrix form. Introducing the matrix notation, we are able not only to simplify expressions for appropriate moments, but also to observe the interplay between mortality and interest rate. Moreover, matrix notation not only makes calculations easier, but also provides a nice form for important equations, as for example equivalence principle (see Remark 2).

In this paper, by the *insurance contract* we mean a general life insurance (as temporary insurance, pure endowment, endowment) or a general annuity policy (as temporary life annuity due, immediate life annuity) with the finite term of policy.

We focus on the discrete-time model, when insurance payments are made at the ends of time intervals.

Note that premium is risk-rated for individuals and groups on the basis of characteristics such as age, sex, occupation and previous health conditions. Also it depends on the largeness of a portfolio. It is more difficult to diffuse risks in small groups (like an individual insurance contract) than in large groups. When a portfolio gets larger, the average expected cash value of payment streams is less likely to vary. Thus we analyze both a single policy and a portfolio of policies.

The paper is organized as follows. A general portfolio of life insurance contracts and assumptions regarding the decrements and the discounting function are presented in Section 2. After a brief description of the cash value of cumulative payment streams generated by insurance contract (Section 3), in Section 4 we derive expressions for the first two moments of the cash value of the stream of future payments. The main result is given in Section 5, where the moments of the cash value of future payment streams are given in a matrix form. Section 6 deals with the analysis of a portfolio of insurance contracts. Selected stochastic models for the rate of interest and results which relieve to define suitable moments of discount function are presented in Section 7. Numerical examples are provided in Section 8. Detailed proofs of the results presented in Sections 5–7 are given in Section 9.

## 2. Decrements

To fix ideas and terminology, consider an insurance contract issued at time 0 and (according to plan) terminating at a later time $n$ ($n$ is the term of policy). Let the age of the insured at the date of issue be $x$. The curtate future-lifetime of the insured with age at entry $x$ is denoted by a random variable $K(x) = K$. Note that using classical actuarial notation (Gerber, 1990; Bowers et al., 1986), we have $\mathbb{P}(K = k) = {}_{k|}q_x$ and $\mathbb{P}(K \geq k) = {}_k p_x$.

Apart from individual insurance contracts we also analyze a portfolio of insurance policies. The size of a portfolio is denoted by $N$ ($N \in \mathbb{N}$) and $K_l$ denotes the curtate future-lifetime of the $l$th insured in the portfolio.