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# Valuation and hedging of life insurance liabilities with systematic mortality risk

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**Abstract**

This paper considers the problem of valuating and hedging life insurance contracts that are subject to systematic mortality risk in the sense that the mortality intensity of all policy-holders is affected by some underlying stochastic processes. In particular, this implies that the insurance risk cannot be eliminated by increasing the size of the portfolio and appealing to the law of large numbers. We propose to apply techniques from incomplete markets in order to hedge and valuate these contracts. We consider a special case of the affine mortality structures considered by Dahl [Dahl, M., 2004. Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. Insurance Math. Econom. 35, 113–136], where the underlying mortality process is driven by a time-inhomogeneous Cox–Ingersoll–Ross (CIR) model. Within this model, we study a general set of equivalent martingale measures, and determine market reserves by applying these measures. In addition, we derive risk-minimizing strategies and mean-variance indifference prices and hedging strategies for the life insurance liabilities considered. Numerical examples are included, and the use of the stochastic mortality model is compared with deterministic models.


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## 1. Introduction

During the past decades, the expected life length has increased considerably in many countries. This has forced life insurers to adjust expectations towards the underlying mortality laws used to determine reserves. Since the future mortality is unknown, a correct description requires a stochastic model, as has already been proposed by several authors; see e.g. Marocco and Pitacco (1998), Milevsky and Promislow (2001), Dahl (2004), Cairns et al. (2004),

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Biffis and Millossovich (2004) and references therein. For a survey of current developments in the literature and their relation to our results, we refer the reader to Section 2 in Dahl (2004). The main contribution of the present paper is not the introduction of a specific model for the mortality intensity, but rather the study of the problem of valuating and hedging life insurance liabilities that are subject to changes in the underlying mortality intensity.

In Dahl (2004), a general class of diffusion models is considered for the mortality intensity, and the affine mortality structures are recognized as a class with particularly nice properties. Here we study a special case of the general affine mortality structures and demonstrate how such models could be applied in practice. Given an initial mortality intensity curve, we assume that the mortality intensity at a given future point in time at a given age is obtained by correcting the initial mortality intensity by the outcome of some underlying process, which is modeled via a time-inhomogeneous Cox–Ingersoll–Ross (CIR) model. Our model implies that the mortality intensity is described by a time-inhomogeneous CIR model as well. As noted by Dahl (2004), the survival probability can now be determined by using standard results for affine term structures.

Within this setting, we consider an insurance portfolio and assume that the individual lifetimes are affected by the same stochastic mortality intensity. In particular, this implies that the lifetimes are not stochastically independent. Hence, the insurance company is exposed to systematic as well as unsystematic mortality risk. Here, as in Dahl (2004), systematic mortality risk refers to the risk associated with changes in the underlying mortality intensity, whereas unsystematic mortality risk refers to the risk associated with the randomness of deaths in a portfolio with fixed mortality intensity. The systematic mortality risk is a non-diversifiable risk, which does not disappear when the size of the portfolio is increased, whereas the unsystematic mortality risk is diversifiable. Since the systematic mortality risk typically cannot be traded efficiently in the financial markets or in the reinsurance markets, this leaves open the problem of pricing insurance contracts. We follow Dahl (2004) and apply financial theories for pricing the contracts. We work with a simple financial market consisting of a savings account and a zero coupon bond. For a particularly simple class of equivalent martingale measures, we derive market reserves for general life insurance liabilities. These market reserves depend on the market's attitude towards systematic and unsystematic mortality risk. Based on an investigation of some Danish mortality data, we propose some pragmatic parameter values and calculate market reserves by solving appropriate versions of Thiele's differential equation.

Furthermore, we investigate methods for hedging and valuating general insurance liabilities in incomplete financial markets. One possibility is to apply risk-minimization, which has been suggested by Föllmer and Sondermann (1986) and applied to insurance risk by Møller (1998, 2001a,c). We demonstrate how risk-minimizing hedging strategies may be determined in the presence of systematic mortality risk. These results generalize the results in Møller (1998, 2001c), where risk-minimizing strategies were obtained for unsystematic mortality risk. As a second possibility we apply utility indifference valuation and hedging, which has gained considerable interest over the last years as a method for valuation and hedging in incomplete markets; see e.g. Schweizer (2001b), Becherer (2003) and references therein. For an application to insurance risk, see Møller (2001b, 2003a,b). We derive mean-variance indifference prices within our model, thus extending the results obtained in Møller (2001b). The hedging and valuation results in this paper are an extension of Dahl (2004), where market reserves were derived without addressing the hedging aspect.

The present paper is organized as follows. Section 2 contains a brief analysis of some Danish mortality data. The general framework is introduced in Section 3. Here, Section 3.1 contains the model for the underlying mortality intensity and the derivation of the corresponding survival probabilities and forward mortality intensities. The financial market used for the calculation of market reserves, hedging strategies and indifference prices is introduced in Section 3.2, and the insurance portfolio is described in Section 3.3. Section 4 presents the class of equivalent martingale measures considered, the insurance payment process and the associated market reserves. Risk-minimizing hedging strategies are determined in Section 5, and mean-variance indifference prices and hedging strategies are obtained in Section 6. Numerical examples are provided in Section 7, and Appendix A contains proofs of some technical results.

## 2. Motivation and empirical evidence

We briefly describe typical empirical findings related to the development in the mortality during the last couple of decades. The results in this section are based on Danish mortality data, which have been compiled and analyzed by Andreev (2002). A more detailed statistical study is carried out in Fledelius and Nielsen (2002), who applied kernel hazard estimation.