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Hedging life insurance with pure endowments

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Abstract

We extend the work of Milevsky et al., [Milevsky, M.A., Promislow, S.D., Young, V.R., 2005. Financial valuation of mortality risk via the instantaneous Sharpe ratio (preprint)] and Young, [Young, V.R., 2006. Pricing life insurance under stochastic mortality via the instantaneous Sharpe ratio (preprint)] by pricing life insurance and pure endowments together. We assume that the company issuing the life insurance and pure endowment contracts requires compensation for their mortality risk in the form of a pre-specified instantaneous Sharpe ratio. We show that the price $P^{m,n}$ for m life insurances and n pure endowments is less than the sum of the price $P^{m,0}$ for m life insurances and the price $P^{0,n}$ for n pure endowments. Thereby, pure endowment contracts serve as a hedge against the (stochastic) mortality risk inherent in life insurance, and vice versa.

Keywords: Stochastic mortality; Sharpe ratio; Life insurance; Pure endowments; Non-linear partial differential equations

1. Introduction

Milevsky et al. (2005) present a framework for valuing (stochastic) mortality risk. They do this by assuming that the company issuing a mortality-contingent claim requires compensation for this risk in the form of a pre-specified instantaneous Sharpe ratio. They apply their method to price pure endowment contracts and show that the resulting price satisfies many desirable properties. Young (2006) applies the same method to price life insurance that pays at the moment of death of the insured.

We extend their work by pricing life insurance and pure endowments together. We show that the price $P^{m,n}$ for m life insurances and n pure endowments is less than the sum of the price $P^{m,0}$ for m life insurances and the price $P^{0,n}$ for n pure endowments. Thereby, pure endowment contracts serve as a partial hedge against the (stochastic) mortality risk inherent in life insurance, and vice versa. In our framework the price of an additional life insurance contract can be defined as the marginal price, $P^{m+1,n} - P^{m,n}$, when the insurer already holds m life insurances and n pure endowments. Similarly, one can marginally price an additional pure endowment. Defining a price in this way is related to work of Stoikov (2005), in which he uses the principle of equivalent utility to find the indifference price of an additional financial contract.

In this paper, we assume that the mortalities of the individuals buying the contracts are independent given the hazard rate, but that all the buyers share the same hazard rate. It is the positive correlation induced by the identical

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hazard rates that causes pure endowments to be a partial hedge against the mortality risk in life insurance. If, instead of assuming they are identical, we were to assume that the hazard rates of the buyers are positively correlated, then we would also obtain that pure endowment serves as a hedge to life insurance.

The remainder of the paper is organized as follows. In Section 2, we present our modeling framework and then show how to price a portfolio of life insurance and pure endowment contracts via the instantaneous Sharpe ratio, the method proposed by Milevsky et al. (2005). In Section 3, we show that the price $P^{m,n}$ for m life insurances and n pure endowments is monotone in m and n. We also show that $P^{m,n} \leq P^{m_1,n_1} + P^{m_2,n_2}$, in which $m_1 + m_2 = m$ and $n_1 + n_2 = n$ (subadditivity). Thus, life insurance and pure endowments are hedges for each other. As a corollary of the subadditivity, we show that the marginal price of a life insurance and pure endowment contract satisfy are in fact reasonable prices. Section 4 concludes the paper.

2. Instantaneous Sharpe ratio

2.1. Stochastic mortality model and financial market

In this section, we assume that an insurer has issued m life insurance and n pure endowment contracts, for $m, n \in \mathbb{N}$. In the setting of this paper, a life insurance contract is an obligation to pay \$1 at time T if the insured dies before that time, and a pure endowment is an obligation to pay \$1 at time T if its holder is alive then. The insurer also trades in the bond market to hedge its interest rate risk. First, we obtain the hedging strategy for the insurance company and then describe how to use the instantaneous Sharpe ratio to find the price of a basket of m life insurance and n pure endowment contracts.

We use the stochastic mortality modeling framework introduced in Milevsky et al. (2005). We assume that the mortalities of the individuals purchasing the contracts are independent given the hazard rate, and we assume that the (common) hazard rate λ follows

$$d\lambda_t = \mu(\lambda_t, t)(\lambda_t - \underline{\lambda})dt + \sigma(t)(\lambda_t - \underline{\lambda})dW_t. \tag{2.1}$$

Here, $\underline{\lambda} > 0$ and W is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$. We need some technical assumptions on the coefficients of this diffusion.

Assumption 2.1. In the rest of the paper we assume that

1. The volatility function σ is either identically zero, or it is a continuous function of t such that $\sigma(t) > \kappa > 0$, $t \in [0, T]$.
2. The drift function μ is a Lipschitz continuous function of λ and t and there exists $\varepsilon > 0$ such that if $0 < \lambda - \underline{\lambda} < \varepsilon$, then $\mu(\lambda, t) > 0$, $t \in [0, T]$.

Under these assumptions a unique strong solution of (2.1) exists, and if the process starts at $\lambda_0 > \underline{\lambda}$, then $\lambda_t > \underline{\lambda}$ for all $t \in [0, T]$.

To determine the value of the basket of m life insurances and n pure endowments, we create a portfolio composed of the obligation to pay these, the pure endowments and life insurances and of default-free zero-coupon bonds that pay \$1 at time T. This requires a model for the bond prices, and we use a short rate model.

The dynamics of the short rate r, the rate at which the money market grows, are given by

$$dr_t = a(r_t, t)dt + b(r_t, t)dB_t, \tag{2.2}$$

in which $a, b \geq 0$ are deterministic functions, and B is a Brownian motion with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, that is independent of W, the Brownian motion driving the hazard rate in (2.1). Note that a natural epidemic, which would shock λ, would also have an impact on the underlying economy, so that the short rate would be affected. Therefore, it would be entirely reasonable to assume that W and B are correlated; however, for simplicity we assume that they are independent.

In this framework, the price of a default-free zero-coupon bond (at time t) that pays \$1 at time T is given by

$$F(r, t; T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s ds\right)\middle| r_t = r\right], \tag{2.3}$$