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Asymmetric confidence bands for simple linear regression over bounded intervals

Walter W. Piegorsch[a,*], R. Webster West[a], Obaid M. Al-Saidy[a], Kelly D. Bradley[b]

[a] *Department of Statistics, University of South Carolina, Columbia, SC 29208, USA*
[b] *School of Educational Policy and Leadership, Ohio State University, Columbus, OH 43210, USA*

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Abstract

Computation of simultaneous confidence bands is described for simple linear regressions where the band is constructed to be asymmetric about the predictor mean. Both two-sided and one-sided bands are constructed. The bands represent extensions of a class of symmetric confidence bands due to Bowden, 1970. J. Amer. Statist. Assoc. 65, 413–421. An example illustrates the computations, and a WWW-based applet for computing the bands is described. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Linear regression confidence bands

In many experimental situations, the response variable, Y, is observed in tandem with a non-stochastic predictor variable, x. Often, the mean response is assumed linear in x: $E[Y] = \beta_0 + \beta_1 x$. A common model for this simple linear regression setting assumes that the observations are sampled from a normal parent distribution:

* Corresponding author. Tel.: +1-803-7777800.
E-mail address: piegorsc@stat.sc.edu (W.W. Piegorsch).

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$Y_i \sim$ indep. $N(\beta_0 + \beta_1 x_i, \sigma^2)$, $i = 1, \ldots, n$. In this case, least-squares estimators $\boldsymbol{b} = [b_0, b_1]'$ of the unknown parameter vector $\boldsymbol{\beta} = [\beta_0, \beta_1]'$ correspond to those achieved under maximum likelihood, and exact inferences on the effects of x or on the mean response are readily available (Neter et al., 1996).

Herein we direct attention at construction of simultaneous $1 - \alpha$ *confidence bands* for the underlying mean $\beta_0 + \beta_1 x$ over all values of x in some relevant restriction set $\mathscr{B}$. A confidence band is best written as a set of parameters. $\mathscr{C} = \{\boldsymbol{\beta}\colon |(b_0 - \beta_0) + (b_1 - \beta_1)x| \leq h_\alpha \ell(x) S,\ \forall x \in \mathscr{B}\}$, where h_α is a critical point that gives the band $1 - \alpha$ coverage, S^2 is the usual mean square error estimator of σ^2 and $\ell(x)$ is the band shape function. In cases where the population variance is known, one simply replaces S^2 with the known value of σ^2 and modifies the value of h_α appropriately (see Section 2, below).

To simplify this notation, let $U_j = (b_j - \beta_j)/S$, $j = 0, 1$. Then, the band's confidence region becomes

$$\mathscr{C} = \{\boldsymbol{U}\colon |U_0 + U_1 x| \leq h_\alpha \ell(x),\ \forall x \in \mathscr{B}\}. \tag{1.1}$$

If one-sided bands are desired, the confidence region may be written simply as

$$\begin{aligned} \mathscr{D} &= \{\boldsymbol{\beta}\colon \beta_0 + \beta_1 x \leq b_0 + b_1 x + k_\alpha \ell(x) S,\ \forall x \in \mathscr{B}\} \\ &= \{\boldsymbol{U}\colon U_0 + U_1 x \geq -k_\alpha \ell(x),\ \forall x \in \mathscr{B}\}, \end{aligned} \tag{1.2}$$

where k_α is a one-sided critical point that again gives the band $1 - \alpha$ coverage. (The display gives upper one-sided bands; lower one-sided bands follow by reversing the set of inequalities and changing the sign on k_α.)

In practice, an interval restriction on x is common: $\mathscr{B} = \{x\colon A \leq x \leq A + B\}$, where A and $B > 0$ are predetermined constants. Interval restrictions have been considered by many authors; see, e.g., Wynn and Bloomfield (1971), Naiman (1983), or Stewart (1991) among many others. In some applications the predictor variable may be a positive quantity such as dose, mass, time, a dietary supplement, etc. In such cases, it is natural to restrict x to be positive, i.e., where $A = 0$ and B becomes some known upper bound on the possible value of x. Below, we operate under this assumption and fix the restriction set as $\mathscr{B} = \{x\colon 0 \leq x \leq B\}$. (Notice, however, that setting $A = 0$ does not sacrifice any generality: when interest exists over the more general interval, simply transform the predictor variable to $x' = x - A$ and then work on the scale we use herein.)

The most widely recognized band is the Working–Hotelling–Scheffé construction (Working and Hotelling, 1929; Scheffé, 1953) which in a simplified form is $\ell(x) = [1 + (x - \bar{x})^2]^{1/2}$. This is in fact a member of a larger class of bands, the Bowden p-family $\ell(x) = [1 + (x - \bar{x})^p]^{1/p}$, where $p \geq 1$ is a shape parameter that determines the bands' degree of incline near $x = \bar{x}$ (Bowden, 1970).

1.2. Asymmetric confidence bands

An important feature of the Bowden family is that its bands are symmetric about $\bar{x}$; i.e., their width is identical at any predictor values equidistant from $\bar{x}$. This quality is tied directly to the fact that the variance of the estimated linear predictor, $b_0 + b_1 x$,