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Valuation of life insurance products under stochastic interest rates

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Abstract

In this paper, we introduce a consistent pricing method for life insurance products whose benefits are contingent on the level of interest rates. Since these products involve mortality as well as financial risks, we present an approach that introduces stochastic models for insurance products through stochastic interest rate models. Similar to Black et al. [Black, Fisher, Derman, Emanuel, Toy, William, 1990. A one-factor model of interest rates and its application to treasury bond options. Financ. Anal. J. 46 (January–February), 33–39], we assume that the premiums and volatilities of standard insurance products are given exogenously. We then project insurance prices to extract underlying martingale probability structures. Numerical examples on variable annuities are provided to illustrate the implementation of this method.

1. Introduction

Most fixed and variable annuities include interest rate guarantees that protect the policyholder against a poor performance of the reference account. The reference account for fixed annuities is the insurance company general account. However, the premiums for variable annuities are invested in the policyholder's choice of underlying stock and/or bond funds, which is called the subaccount.[1] Variable annuity financial guarantees are known as the guaranteed minimum death benefit (GMDB), the guaranteed minimum accumulation benefit (GMAB), the guaranteed minimum income benefit (GMIB), and/or the guaranteed minimum surrender benefit (GMSB). The last three benefits refer to guaranteed minimum living benefits (GMLB). See the monograph by Hardy (2003) for comprehensive discussions on these guarantees. According to the 2006 Annuity Fact Book (see National Association for Variable Annuities (2006)), the total sales in 2005 for fixed and variable annuities have reached $212.3 billion in the US.

The traditional insurance and annuity pricing method calculates the net premium of a product as the expected present value of its benefits with respect to a mortality law. The (gross) premium, net of commissions and other non-mortality-related charges, is determined as the net premium plus a loading that is based on a certain premium principle (see Bowers et al. (1997)). Unfortunately, this traditional actuarial pricing approach often produces premiums inconsistent with the insurance market. Furthermore, it is difficult to extend to the valuation of non-standard insurance products since these products are embedded with various types of financial guarantees and are more sensitive to interest rates. Many attempts have been made to evaluate insurance products that are linked to the financial market using option pricing and stochastic mortality. However, most of the developments have been made in a fully continuous-time setting under an independence assumption between insurance and interest rates. Milevsky and Promislow (2001) assume that the force of mortality has a mean reverting Brownian Gompertz to evaluate mortality-contingent claims. Dahl (2004), Biffis and Millossovich (2006), and Biffis (2005) use affine mortality processes to evaluate insurance products that are linked to the financial market. Cairns et al. (2006) develop frameworks for the force of mortality based on interest rate models. In the discrete-time framework, Lee (2000) applies a stochastic adjustment to deterministic mortality probabilities. Lin and Cox (2005) adjust the mortality probabilities using the Wang Transform.

In this paper, we propose a market consistent valuation method for insurance products that contain financial guarantees, using the independence assumption as well as the conditional independence assumption between the policyholder and the interest rates. Similar to Jarrow and Turnbull (1995), we

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[1] We focus on subaccounts of variable annuities that are invested in the fixed income market.

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derive martingale probability measures associated with basic insurance products and specifically the term life insurances and pure endowment insurances. We assume that the premium[2] information of term life insurances and pure endowment insurances at all maturities is obtainable. Two martingale measures, each of which serves a different purpose, are derived using the premium information for each insurance product and assumption. Consequently, these measures are age dependent, include an adjustment for the mortality risk, and reproduce premiums of the respective insurance products. Similar to Black et al. (1990), we assume that the volatilities for standard insurance products are given exogenously under the conditional independence assumption. We then project insurance prices to extract underlying martingale probability structures. For implementation purposes additional structure is also proposed to find implied volatilities. We implement these approaches by evaluating the costs of interest rate guarantees embedded in variable annuities, when their subaccounts are invested in fixed income securities.

The assumption that the premium information of term life insurances and pure endowment insurances is available is what differs from many studies of the valuation of non-standard insurance products. We argue that this assumption is reasonable because an insurance company that issues insurance products containing financial guarantees often issues standard insurances. The premium information and fee structure of these products are available within the company and hence it may be possible, although might not be easy, for a valuation actuary in the company to extract the aforementioned premium information by stripping the costs of any add-on conversion options and riders and by comparing the premiums of insurances or annuities with different maturities. Furthermore, due to the relative efficiency of the current insurance market and the standardization of these basic products, their premium are often dictated by the market and hence vary insignificantly from company to company. As a result, the proposed method in this paper will provide market consistent values for insurance products with financial guarantees and will reduce subjectivity when determining loadings.

This paper is organized as follows. The next section presents binomial models for the financial market as well as for the term life insurance and pure endowment insurance products. We then derive martingale measures for those standard insurance products in Section 3 under the independence and conditional independence assumptions. It is followed by numerical examples presenting the corresponding martingale probabilities. Finally, we examine the implications of the proposed approaches on guaranteed benefits embedded in variable annuity by conducting a detailed numerical analysis.

2. Underlying binomial models

In this section, we first introduce the BDT binomial model (Black et al. (1990)) for short-term rates. Among other advantages, the BDT model is used by practitioners because it matches the current term structure of interest rates and the volatilities. A detailed analysis may be found in Panjer et al. (1998) or Lin (2006). We then introduce two binomial insurance models: one for term life insurances and one for pure endowment insurances.

2.1. The Black, Derman and Toy model

In the BDT model, the short-term rate over a year either goes up or down. In order to facilitate computation, the tree is recombining and the short-term rate is Markovian. At year t, the short-term rate can take exactly $t+1$ distinct values denoted by $r(t,0), r(t,1), \ldots, r(t,t)$. Indeed, $r(t,l)$ represents the short-term rate between time t and $t+1$ that has made "l" up moves. Specifically, the short-term rate today, $r(0)$ is equal to $r(0,0)$, and in the case where $r(t)=r(t,l)$, the short-term rate at time $t+1$, $r(t+1)$ can only take two values, either $r(t+1,l)$ (decrease) or $r(t+1,l+1)$ (increase). We consider the short-term rate process under the martingale measure Q and hence, the discounted value process $L(t,T)/B(t)$ is a martingale. $L(t,T)$ represents the price at time t of a default-free, zero-coupon bond paying one monetary unit at time T and $B(t)$, the money market account, represents one monetary unit ($B(0)=1$) accumulated at the short-term rate

$$B(t)=\prod_{i=0}^{t-1}[1+r(i)]. \tag{1}$$

Let $q(t,l)$ be the probability under Q that the short-term rate increases at time $t+1$ given $r(t)=r(t,l)$. That is

$$q(t,l)=Q[r(t+1)=r(t+1,l+1)|r(t)=r(t,l)], \tag{2}$$

for $0\le l\le t$, which is set to be 0.5 under the BDT model. Consequently, the martingale probability that the short-term rate decreases at time $t+1$ given $r(t)=r(t,l)$ is also 0.5. Fig. 1 describes the dynamic of the short-term rate process.

We assume that the model matches an array of yield volatilities $(\sigma_r(1), \sigma_r(2), \ldots)$, which is assumed to be observable from the financial market. This vector is deterministic, specified at time 0, and each element is defined by

$$\sigma_r(t)^2 = \mathrm{Var}\,[\ln r(t)|r(t-1)=r(t-1,l)] = \left[0.5\ln\left(\frac{r(t,l+1)}{r(t,l)}\right)\right]^2, \tag{3}$$

for $l=0,1,\ldots,t-1$ and $t=1,\ldots$. Hence, $r(t,l+1)$ is larger than $r(t,l)$ thus, (3) may be rewritten as follows

$$\sigma_r(t)=0.5\ln\left(\frac{r(t,l+1)}{r(t,l)}\right). \tag{4}$$

Eq. (4) holds for $l\in\{0,1,\ldots,t-1\}$ and leads to

$$r(t,l)=r(t,0)^{1-l/t}r(t,t)^{l/t}, \tag{5}$$

for $l=0,1,\ldots,t$. Eqs. (4) and (5) lead to

$$\sigma_r(t)=\frac{1}{2t}\ln\left(\frac{r(t,t)}{r(t,0)}\right). \tag{6}$$

[2] Hereafter, a premium is referred to as a single premium that includes mortality charges but excludes commissions and other non-mortality-related charges.