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Pricing life insurance under stochastic mortality via the instantaneous Sharpe ratio

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Abstract

We develop a pricing rule for life insurance under stochastic mortality in an incomplete market by assuming that the insurance company requires compensation for its risk in the form of a pre-specified instantaneous Sharpe ratio. Our valuation formula satisfies a number of desirable properties, many of which it shares with the standard deviation premium principle. The major result of the paper is that the price per contract solves a *linear* partial differential equation as the number of contracts approaches infinity. One can represent the limiting price as an expectation with respect to an equivalent martingale measure. Via this representation, one can interpret the instantaneous Sharpe ratio as a market price of mortality risk. Another important result is that if the hazard rate is stochastic, then the risk-adjusted premium is greater than the net premium, even as the number of contracts approaches infinity. Thus, the price reflects the fact that systematic mortality risk cannot be eliminated by selling more life insurance policies. We present a numerical example to illustrate our results, along with the corresponding algorithms.

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1. Introduction

We propose a pricing rule for life insurance when interest rates and mortality rates are stochastic by applying the method developed and expounded upon by Milevsky et al. (2005, 2007). In the case addressed in this paper, their method amounts to targeting a pre-specified Sharpe ratio for a portfolio of bonds that optimally hedges the life insurance, albeit only partially.

Actuaries often assume that one can eliminate the uncertainty associated with mortality by selling a large number of insurance contracts. This assumption is valid if the force of mortality is deterministic. Indeed, if the insurer sells enough contracts, then the average deviation of actual results from what is expected goes to zero, so the risk is diversifiable. However, because the insurer can only sell a finite number of insurance policies, it is impossible to eliminate the risk that experience will differ from what is expected. The risk associated with selling a finite number of insurance contracts is what we call the *finite portfolio risk*.

On the other hand, if the force of mortality for a population is stochastic, then there is a systematic (that is, common) risk that cannot be eliminated by selling more policies. We call this risk the *stochastic mortality risk*, a special case of stochastic parameter risk. Even as the insurer sells an arbitrarily large number of contracts, the systematic stochastic mortality risk remains.

We argue that mortality is uncertain and that this uncertainty is correlated across individuals in a population – mostly due to medical breakthroughs or environmental factors that affect the entire population. For example, if there is a positive probability that medical science will find a cure for cancer during the next thirty years, this will influence aggregate mortality patterns. Biffis (2005), Schrager (2006), Dahl (2004), as well as Milevsky and Promislow (2001), use diffusion processes to model the force of mortality, as we do in this paper. One could model catastrophic events that affect mortality

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widely, such as epidemics, by allowing for random jumps in the force of mortality.

In related work, Blanchet-Scalliet et al. (2005) value assets that mature at a random time by using the principle of no arbitrage by focusing on equivalent martingale measures; the resulting pricing rules are, therefore, linear. Dahl and Møller (2006) take a similar approach in their work. However, for insurance markets, one cannot assert that no arbitrage holds, so we use a different method to value life insurance contracts and our resulting pricing rule is non-linear, except in the limit.

We value life insurance by assuming that the insurance company is compensated for its risk via the so-called *instantaneous Sharpe ratio* of a suitably-defined portfolio. Specifically, we assume that the insurance company picks a target ratio of expected excess return to standard deviation, denoted by α, and then determines a price for a life insurance contract that yields the given α for the corresponding portfolio. One might call α the market price of mortality risk, but it appears in the pricing equation in a non-linear manner. However, as the number of life insurance policies increases to infinity, then this α is the market price of risk in the "traditional" sense of pricing in financial markets in that it acts to modify the drift of the hazard rate process.

In this paper, we assume that the life insurance company does not sell annuities to hedge the stochastic mortality risk. In related work, Bayraktar and Young (2007a) allow the insurer to hedge its risk partially by selling pure endowments to individuals whose stochastic mortality is correlated with that of the buyers of life insurance.

We obtain a number of results from our methodology that one expects within the context of insurance. For example, we prove that if the hazard rate is deterministic, then as the number of contracts approaches infinity, the price of life insurance converges to the net premium under the physical probability for mortality and the risk neutral probability for interest rates. In other words, if the stochastic mortality risk is not present, then the price for a large number of life insurance policies reflects this and reduces to the usual expected value pricing rule in the limit. Alternatively, one can see that in the limit the average cash flow is certain, hence, the price becomes that of a zero-coupon bond with a rate of discount modified to account for the rate of dying.

An important theorem of this paper is that as the number of contracts approaches infinity, the limiting price per risk solves a *linear* partial differential equation and can be represented as an expectation with respect to an equivalent martingale measure as in Blanchet-Scalliet et al. (2005). Therefore, we obtain their results as a limiting case of ours. Moreover, if the hazard rate is stochastic, then the value of the life insurance contract is greater than the net premium, even as the number of contracts approaches infinity. Milevsky et al. (2005) obtain similar results when pricing pure endowments.

The remainder of this paper is organized as follows. In Section 2, we present our financial market, describe how to use the instantaneous Sharpe ratio to price life insurance payable at the moment of death, and derive the resulting partial differential equation (pde) that the price $A = A^{(1)}$ solves. We also present the pde for the price $A^{(n)}$ of n conditionally independent and identically distributed life insurance risks. In Section 3, we study properties of $A^{(n)}$; our valuation operator is subadditive and satisfies a number of other appealing properties. In Section 4, we find the limiting value of $\frac{1}{n}A^{(n)}$ and show that it solves a linear pde. We also decompose the risk charge for a portfolio of life insurance policies into a systematic component (due to uncertain aggregate mortality) and a non-systematic component (due to insuring a finite number of policies). In Section 5, we present a numerical example that illustrates our results, along with the corresponding algorithms that we use in the computation. Section 6 concludes the paper.

2. Instantaneous Sharpe ratio

In this section, we describe the term life insurance policy and present the financial market in which the issuer of this contract invests. We obtain the hedging strategy for the issuer of the life insurance. We describe how to use the instantaneous Sharpe ratio to price the insurance policy, and we derive the resulting partial differential equation (pde) that the price solves; see Eq. (2.16). We also present the pde for the price $A^{(n)}$ of n conditionally independent and identically distributed life insurance risks; see Eq. (2.19).

2.1. Mortality model and financial market

We use the stochastic model of mortality of Milevsky et al. (2005). We model the hazard rate for an individual or set of individuals of a given age. If we were to model a population's hazard rate, then we would take into account age and time as in Lee and Carter (1992) and, more recently, Ballotta and Haberman (2006). However, because we consider a single age, we simply model the hazard rate as a stochastic process over time.

We assume that the hazard rate λ_t (or force of mortality) of an individual at time t follows a diffusion process such that if the process begins at $\lambda_0 > \underline{\lambda}$ for some positive constant $\underline{\lambda}$, then $\lambda_t > \underline{\lambda}$ for all $t \geq 0$. From a modeling standpoint, $\underline{\lambda}$ could represent the lowest attainable hazard rate remaining after all causes of death such as accidents and homicide have been eliminated; see, for example, Gavrilov and Gavrilova (1991) and Olshansky et al. (1990).

Specifically, we assume that

$$d\lambda_t = \mu(\lambda_t, t)(\lambda_t - \underline{\lambda})dt + \sigma(t)(\lambda_t - \underline{\lambda})dW_t^{\lambda}, \tag{2.1}$$

in which $\{W_t^{\lambda}\}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbf{P})$. The volatility σ is either identically zero, or it is a continuous function of time t bounded below by a positive constant κ on $[0, T]$. The drift μ is a Hölder continuous function of λ and t for which there exists $\epsilon > 0$ such that if $0 < \lambda - \underline{\lambda} < \epsilon$, then $\mu(\lambda, t) > 0$ for all $t \in [0, T]$. After Lemma 3.2, we add more requirements for μ. Note that if $\sigma \equiv 0$, then λ_t is deterministic, and in this case, we write $\lambda(t)$ to denote the deterministic hazard rate at time t.

Suppose an insurer issues a term life insurance policy to an individual that pays \$1 at the moment of death if the individual