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# Optimal investment and life insurance strategies under minimum and maximum constraints

Peter Holm Nielsen[a,*], Mogens Steffensen[b]

[a] PFA Pension, Sundkrogsgade 4, DK-2100 Copenhagen, Denmark
[b] Department of Applied Mathematics and Statistics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

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**Abstract**

We derive optimal strategies for an individual life insurance policyholder who can control the asset allocation as well as the sum insured (the amount to be paid out upon death) throughout the policy term. We first consider the problem in a pure form without constraints (except nonnegativity on the sum insured) and then in a more general form with minimum and/or maximum constraints on the sum insured. In both cases we also provide the optimal life insurance strategies in the case where risky-asset investments are not allowed (or not taken into consideration), as in basic life insurance mathematics. The optimal constrained strategies are somewhat more complex than the unconstrained ones, but the latter can serve to ease the understanding and implementation of the former.


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## 1. Introduction

This paper is concerned with optimal strategies regarding life insurance (i.e., coverage against death) and investment for an individual policyholder in a life insurance company or a pension fund (referred to as the *company* henceforth). More specifically, we consider a life insurance policy comprising life insurance as well as retirement saving during $[0, T]$, where $T > 0$ is a fixed finite time horizon. The policyholder is allowed to choose, in a continuous manner, the *sum insured*, which is the sum to be paid out from the company upon death of the policyholder, as well as the investment strategy. Our main focus is on the issue of optimal life insurance rather than optimal investment (although, as we shall see, these issues should not be viewed separately), and we therefore work throughout with a simple and well-known model for the financial market offering the assets available for investment.

We impose in general the very realistic constraint that the sum insured must be nonnegative, but as the title of the paper suggests we shall also consider the optimization problem under more restrictive minimum and maximum constraints in the form of lower and upper boundaries for the sum insured. The motivation for the company to impose an upper boundary is rather obvious, since it puts a limit on the company's immediate risk at any time during the policy term, but the motivation for a lower boundary may not be obvious. However, a lower boundary is sometimes imposed e.g. in pension schemes that are mandatory for employees within a certain line of business in order to ensure that a minimum coverage against death is provided automatically, i.e., without a specific request from each employee.

* Corresponding author. Tel.: +45 39175876; fax: +45 39175952.
*E-mail addresses:* pei@pfa.dk (P.H. Nielsen), mogens@math.ku.dk (M. Steffensen).

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The unconstrained case, i.e., without minimum or maximum constraints (except nonnegativity), is a special case of the constrained case since the lower and upper boundaries can be set to 0 and $\infty$, respectively. However, for educational purposes, and to ease the overall presentation of the results, we have chosen to include the solution of the unconstrained case separately (in Section 2), also because the results from the unconstrained case play an important role as convenient references in the constrained case. The solution of the unconstrained case in itself is not a main contribution of this paper to the literature, though, since it, from a mathematical perspective, actually is equivalent to a certain purely financial consumption/investment problem with a well-known solution (as will be noted).

From a mathematical point of view it is interesting to note that the unconstrained case can be solved by dynamic programming (as is done in this paper), whereas the more general constrained case is quite troublesome (at least) to solve by this approach. The latter case is thus substantially facilitated by the martingale methodology, emphasizing one of the major strengths of this technique, namely that it can lead to (more or less explicit) solutions in problems with binding constraints, where the dynamic programming approach typically is not easy to apply.

An interesting aspect of the optimal insurance strategy in the unconstrained case is that although the optimal sum insured depends heavily on the development of the financial market, its range is the entire interval $[0, \infty)$ at any time during the policy term. Thus, the boundaries imposed in the constrained case are strictly binding whenever they are non-trivial (i.e., strictly positive and finite, respectively).

In general the investment strategy is taken to be unconstrained (except for technical conditions), i.e., we allow all positions in the risky assets. However, we also provide the optimal life insurance strategy in the special case of a market without risky assets, or equivalently, under the constraint that no risky-asset investments (long or short) are allowed. There are two main motivations for this "sub-problem": Firstly, it is more in line with basic life insurance mathematics, where the interest rate is typically assumed constant (or deterministic), see e.g. Møller and Steffensen (2007), so it constitutes an interesting problem in its own right, at least from an actuarial perspective. Various alternative interpretations of the interest rate (which in the general setup below is the risk-free money market rate) are then possible; in particular it may play the role of the so-called *second order rate* or *bonus rate*, see Møller and Steffensen (2007). Secondly, the optimal life insurance strategy as such stands out more clearly and is thus perhaps easier to interpret and analyze. However, we do not provide detailed proofs of our results pertaining to this case (the proofs are similar to the proofs provided in the general setup; the details are left to the interested reader).

Studying the optimal demand for life insurance for an economic agent dates back to Yaari (1965) and has been followed up by Richard (1975), who was the first to study the combined problem of optimal life insurance and investment (and consumption as well), Campbell (1980), and others. The problem variations studied in the literature concern whether some or all of the processes regarding investment, life insurance, and consumption are considered as decision processes, whether the agent has non-capital (wage) income, and whether the problems are solved in discrete or continuous time. More recent contributions to this body of literature are provided by Chen et al. (2006), who allow for stochastic income, Hong and Ríos-Rull (2007), who take a family point of view and also take social security into account, and Ye and Pliska (2007), who study a problem close to the one studied by Richard (1975) and also provide a nice survey of the literature. The main contribution of the present paper is the solution of the continuous-time problem where all processes are decision processes (as in Richard (1975)), and with constraints on the life insurance decision. To the knowledge of the authors, the problem with such constraints has not been considered previously in the literature. Dynamic utility optimization is studied recently in the context of non-life insurance by Moore and Young (2006). Another related body of research concerns optimal investment with the objective of minimizing the lifetime ruin probability (and generalizations); this constitutes a relevant personal finance problem in the absence of life insurance, see e.g. Bayraktar and Young (2007).

The remainder of the paper is organized as follows: Section 2 introduces the general setting and the basic optimization problem, and the unconstrained case is treated. In Section 3 we solve the problem in the general constrained case; this section thus contains the main results of the paper. Section 4 concludes.

Some basic notations: All vectors are column vectors. The transposed of a vector or matrix $a$ is denoted by $a'$. The $d$-dimensional vector of 1's is denoted by $1_d$.

## 2. Setup and basic problem

To formalize the setup we take as given some underlying probability space $(\Omega, \mathcal{F}, \mathrm{P})$, on which all random variables introduced in the following are defined.

We consider a policyholder with a life insurance policy issued at time 0 and terminated at a fixed finite time horizon $T > 0$. Let $\tau$ be a nonnegative random variable representing the (random) time of death of the policyholder. For $t \in [0, T]$, the mortality intensity of $\tau$ is given by a continuous function $\mu : [0, T] \to [0, \infty)$, which means that

$$\mathrm{P}(\tau \leq t) = 1 - \mathrm{e}^{-\int_0^t \mu(s)\mathrm{d}s}, \quad t \in [0, T].$$

The (conditional) distribution of $\tau$ on $(\tau > T)$ is irrelevant in this paper.

Let $W = (W_1, \ldots, W_d)'$ be a $d$-dimensional standard Brownian Motion ($d \in \mathbb{N}$) stochastically independent of $\tau$. The financial market is assumed to be frictionless and to consist of a risk-free money market account with price dynamics given by

$$\mathrm{d}S_0(t)/S_0(t) = r\mathrm{d}t,$$

where $r \geq 0$ is a fixed constant, and $d$ risky assets with price dynamics given by

$$\mathrm{d}S_i(t)/S_i(t) = \alpha_i \mathrm{d}t + \sum_{j=1}^{d} \sigma_{ij}\mathrm{d}W_j(t), \quad i = 1, \ldots, d,$$