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A sensitivity analysis concept for life insurance with respect to a valuation basis of infinite dimension

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Abstract

A sensitivity analysis concept is introduced for prospective reserves of individual life insurance contracts as deterministic mappings of the actuarial assumptions interest rate, mortality probability, disability probability, etc. Upon modeling these assumptions as functions on a real time line, the prospective reserve is here a mapping with infinite dimensional domain. Inspired by the common idea of interpreting partial derivatives of first order as local sensitivities, a generalized gradient vector approach is introduced in order to allow for a sensitivity analysis of the prospective reserves as functionals on a function space. The capability of the concept is demonstrated with an example.

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1. Introduction

By statute the insurer must currently maintain a reserve to meet future liabilities in respect of its insurance contracts. A major concern of regulators and insurers is the choice of the valuation basis, which is the set of assumptions underlying the calculation of premiums and reserves. In practice the true values of future interest or mortality rates are not perfectly known; especially in recent years, financial markets have experienced increased volatility, and life expectancies have risen in many developed countries with an unforeseen rate.

Therefore, the dependence of the reserve on the elements of the valuation basis was always an important issue. Already Lidstone (1905) studied in a discrete time setting the effect on reserves of changes in valuation basis and contract terms, but dealt only with simple single life policies with payments dependent on survival and death. Norberg (1985) transferred Lidstone's ideas to a continuous time version, using Thiele's differential equations. On the same basis, Hoem (1988), Ramlau-Hansen (1988), and Linnemann (1993) obtained a handful of further results. All of these studies yield only qualitative results. They show which direction the prospective reserve or the premium level are shifted to by a parameter change, but do not quantify the magnitude of that effect.

Another approach is to calculate different scenarios and to compare them to each other (see for example Olivieri (2001) or Khalaf-Allah et al. (2006)), but this idea works only for a very small number of parameters.

A third way is to study sensitivities by means of derivatives, which turned out to be a very efficient concept. References using such an approach are Dienst (1995), Bowers et al. (1997), Kalashnikov and Norberg (2003), and Helwich (2003):

Dienst (1995, pp. 66–68, 147–150) uses a finite number of partial derivatives of the net premium with respect to time-discrete disablement probabilities to approximate the relative change of the net premium caused by altered disablement probabilities.

Bowers et al. (1997, pp. 490, 491) calculate the first-order derivative of the expected loss with respect to the interest rate, which they assume to be constant.

Kalashnikov and Norberg (2003) differentiate the prospective reserve and the premium level with respect to one arbitrary

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real parameter, which also includes parameters such as contract terms. They present a ‘dynamical approach’ that allows one to calculate sensitivities even when the prospective reserve is not a closed form expression but given by a differential equation. In Section 5, Kalashnikov and Norberg generalize their approach to a finite number of real parameters.

Helwich (2003), models the actuarial assumptions as finite dimensional and real-valued vectors, allowing for parameter changes at a finite number of discrete time points. He calculates the gradient of the expected loss of a portfolio of insurance contracts with respect to yearly constant interest and retirement rates.

All of those studies have in common that they only consider a finite number of parameters. The present paper introduces a sensitivity analysis based on some generalized gradient vectors. This allows one to study sensitivities with respect to an infinite number of parameters, which meets, for example, the more realistic idea of actuarial assumptions (e.g., mortality) being functions on a real line rather than on a discrete time grid. Nonetheless, the approach includes also discrete time models and thus generalizes Helwich’s (2003) chapter 5.

Concepts for defining ‘functional sensitivities’ are already known in the literature (for example, see Saltelli et al. (2000), chapter 5.7). They can be applied to prospective reserves as functionals of the valuation basis if intensities exist; however, this has not been done so far. In order to jointly comprise the ‘discrete method’ and the ‘continuous method’ of insurance mathematics as well as intermediate cases, Milbrodt and Stracke (1997) proposed a life insurance modeling framework that is based on cumulative intensities. In order to allow the study of sensitivities of prospective reserves with respect to cumulative interest and transition intensities, the present paper extends in Section 3 the functional sensitivity analysis approaches mentioned above. A generalized gradient vector concept for functionals of right-continuous functions with finite total variation is introduced. Similar notions can be found in nonparametric locally asymptotic statistics; differences and similarities are discussed at the end of Section 3.

In Section 4, the general concept of Section 3 is applied on prospective reserves of life insurance contracts as mappings of the valuation basis. The proofs are quite technical and are therefore placed in the Appendix. Section 5 gives an example in order to demonstrate the introduced sensitivity analysis concept.

2. The Markov model

Consider an insurance policy that is issued at time 0, terminates at a fixed finite time T, and is driven by a Markovian jump process $(X_t)_{t\in[0,T]}$ with finite state space $\mathcal{S}$. Let $J := \{(j,k) \in \mathcal{S}^2 | j \neq k\}$ denote the transition space. Following Milbrodt and Stracke (1997), the Markov chain is assumed to possess cumulative intensities of transition denoted by $q_{jk}(t)$, $(j,k) \in J$. According to Andersen et al. (1991, Section II.6), the transition probability matrix

$$p(s,t) := (P(X_t = k | X_s = j))_{(j,k)\in\mathcal{S}^2}, \quad 0 \le s \le t \le T,$$

has the representation

$$p(s,t) = \prod_{(s,t]} (\mathbb{I} + \mathrm{d}q) := \lim_{\max|t_i - t_{i-1}|\to 0} \prod (\mathbb{I} + q(t_i) - q(t_{i-1})), \tag{2.1}$$

for partitions $s < t_0 < t_1 < \cdots < t_n = t$ if the q_{jk} are nondecreasing cadlag (right-continuous with left-hand limits) functions, zero at time zero, $q_{jj} = -\sum_{k\neq j} q_{jk}$, and $\Delta q_{jj}(t) \ge -1$ for all t. The traditional Markov model dates back to a seminal paper of Hoem (1969), where the existence of transition intensities $\mu_{jk}(t)$, $(j,k) \in J$, is assumed and the discounting function is differentiable. In contrast, the approach of Milbrodt and Stracke (1997) makes it possible to jointly comprise the ‘discrete method’ and the ‘continuous method’ of insurance mathematics as well as intermediate cases.
Payments between insurer and policyholder are of two types:

(a) Lump sums are payable upon transitions $(j,k) \in J$ between two states and are specified by deterministic functions $b_{jk} \in BVC$ (*B*ounded *V*ariation on *C*ompacts, see Appendix A.1). To distinguish between the time of transition and the actual time of payment, Milbrodt and Stracke (1997) introduced an increasing function $DT : (0,\infty) \to (0,\infty)$, $DT(t) \ge t$, such that upon transition from j to k at time t the amount $b_{jk}(t)$ is payable at time $DT(t)$. To simplify notation assume that $DT(T) = T$.
(b) Annuity payments fall due during sojourns in a state and are defined by deterministic functions B_j, $j \in \mathcal{S}$, which for each time $t \ge 0$ specify the total amount $B_j(t)$ paid in $[0,t]$. Assume that each B_j is right-continuous and of bounded variation on compacts ($BVC_{\leftarrow}$, cf. Appendix A.1).

Assume that the investment portfolio of the insurance company bears interest with cumulative intensity

$$\Phi \in BVC_{\leftarrow} \quad \text{with } \Delta\Phi(t) := \Phi(t) - \Phi(t-) \ge -C > -1 \quad \text{for all } t \ge 0. \tag{2.2}$$

Thus, the value at time s of a unit payable at time $t > s$ is

$$v(s,t) := \left(\prod_{(s,t]} (1 + \mathrm{d}\Phi)\right)^{-1} = \mathrm{e}^{-(\Phi^c(t) - \Phi^c(s))} \prod_{\tau\in(s,t]} (1 + \Delta\Phi(\tau))^{-1} \tag{2.3}$$

(cf. (2.6.2) in Andersen et al. (1991)), where $\Phi^c(t) := \Phi(t) - \sum_{\tau\le t} \Delta\Phi(\tau)$ for all t.

Remark 2.1. For the existence and the nonnegativity of the discounting factor it would be sufficient to assume that $\Delta\Phi(t) > -1$. But without the lower jump bound $-C > -1$ the discounting factor could rise arbitrarily high by just one big jump. That means that the sensitivity of v to changes of Φ is unlimited, which conflicts with some boundedness condition that will be needed later on.

Let $\Phi = \Phi_+ - \Phi_-$ be the Jordan–Hahn decomposition of Φ into a difference of two nondecreasing functions. From (2.3)