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# On the optimal product mix in life insurance companies using conditional value at risk

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ABSTRACT

This paper proposes a Conditional Value-at-Risk Minimization (CVaRM) approach to optimize an insurer's product mix. By incorporating the natural hedging strategy of Cox and Lin (2007) and the two-factor stochastic mortality model of Cairns et al. (2006b), we calculate an optimize product mix for insurance companies to hedge against the systematic mortality risk under parameter uncertainty. To reflect the importance of required profit, we further integrate the premium loading of systematic risk. We compare the hedging results to those using the duration match method of Wang et al. (forthcoming), and show that the proposed CVaRM approach has a narrower quantile of loss distribution after hedging—thereby effectively reducing systematic mortality risk for life insurance companies.



## 1. Introduction

Over the past decade, a longevity shock has spread across human society. Benjamin and Soliman (1993), McDonald et al. (1998), Grundl et al. (2006) and Stallard (2006) confirm that unprecedented improvements in population longevity have occurred worldwide. The decreasing trend in the mortality rate has created a great risk for insurance companies. The existing literature has proposed a number of solutions to mitigate the threat of longevity risk to life insurance companies. These solutions can be classified into three categories. The *capital market solutions* include mortality securitization (see, for example Dowd, 2003; Lin and Cox, 2005; Cairns et al., 2006a; Blake et al., 2006a,b; Cox et al., 2006), survivor bonds (e.g. Blake and Burrows, 2001; Denuit et al., 2007), and survivor swaps (e.g. Dowd and Blake, 2006; Dowd et al., 2006). These studies suggest that insurance companies can transfer their exposures to the capital markets. Cowley and Cummins (2005) provide an excellent overview of the securitizations of life insurance assets and liabilities. The second set of solutions, the *industry self-insurance solutions,* include the natural hedging strategy of Cox and Lin (2007), the duration matching strategy of Wang et al. (forthcoming), and the reinsurance swap of Lin and Cox (2005). The advantages of these solutions are that the hedging does not require a liquid market and can be arranged at a lower transaction cost. Insurance companies can hedge longevity risk by themselves or with counterparties. The third kind of solution, known as *mortality projection improvement*, provides a more accurate estimation of mortality processes. As Blake et al. (2006b) propose, these studies fall into two areas: continuous-time frameworks (e.g. Milevsky and Promislow (2001), Dahl (2004), Biffis (2005), Schrager (2006), Dahl and Moller (2006)) and discrete-time frameworks, e.g., Brouhns et al. (2002), Renshaw and Haberman (2003) and Cairns et al. (2006b). Parameter uncertainty and model specification in relation to the mortality process have also attracted more attention in recent years.

Among the industry self-insurance solutions, the *natural hedging strategy* suggests that life insurance can serve as a hedging vehicle against longevity risk for annuity products. Wang et al. (forthcoming) employ duration as a measure of the product sensitivity to mortality change, and propose a *mortality duration matching* (MDM) *approach* to calculate the optimal product mix. Their work, however, is based on several restrictive assumptions. First, they assume that future mortality changes involve parallel shifts in the mean, and do not measure the higher-order moments of the mortality risk distribution. Second, the MDM approach applies to only two products. Third, the MDM approach is a pure risk-reduction method because the profit loading is not considered during the hedging procedure. Fourth, Cairns (2000), Melnikov and

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Romaniuk (2006) and Koissi et al. (2006) suggest that parameter risk is crucial when dealing with longevity risk. The parameter uncertainty does not play a role in the MDM approach, since Wang et al. (forthcoming) consider the mortality shift only in terms of its mean.

To overcome these problems, we employ the two-factor stochastic mortality model of Cairns et al. (2006b) and construct the Conditional Value-at-Risk Minimization (CVaRM) (Dowd and Blake, 2006) approach to control the possible loss. Managing products risk with parameter uncertainty is one feature of the CVaRM approach. The other feature is that we add the profit-loading constraint into the optimization. The premium-pricing principle suggested by Milevsky et al. (2006) is employed to estimate the required profit loadings, i.e., in order to compensate the stockholders bearing systematic mortality risk with the same Sharpe ratio as other asset classes in the economy.[1] Furthermore, the CVaRM approach could be easily implemented using linear programming (Rockafellar and Uryasev, 2000), and insurance companies could adopt it as their own internal risk-management tool.

The results of our simulation reveal that the proposed CVaRM approach yields a less dispersed product distribution after hedging and so effectively reduces systematic mortality risk for life insurance companies. The MDM approach, on the other hand, has a limited effect on the dispersion of the product distribution. In addition, the CVaRM approach considers not only risk reduction but also the required profit constraint. We found that the required loading substantially changes the optimal product mix and so cannot be ignored.

The remainder of this article is organized as follows: Section 2 outlines the models, including the mortality model of Cairns et al. (2006b), the duration matching model of Wang et al. (forthcoming), the loading estimation of Milevsky et al. (2006) and the CVaRM approach. In Section 3 we introduce the mortality data, project future mortality and design the products. Section 4 presents the numerical examples in two scenarios: the two-product scenario without a required loading constraint and a multiple-product scenario with a required loading constraint. The hedging results of the MDM and CVaRM approaches are also compared in this section. Conclusions and implications are contained in Section 5.

## 2. The models

Before introducing the CVaRM approach, this section first briefly reviews the two-factor stochastic mortality model of Cairns et al. (2006b), the mortality duration matching model of Wang et al. (forthcoming), and the loading-estimation methods of Milevsky et al. (2006).

### 2.1. The two-factor stochastic mortality model

Several stochastic models proposed in the literature attempt to capture the mortality processes. We chose the two-factor mortality model, i.e., CBD model, as the underlying mortality process for two reasons. First, the CBD model characterizes not only a cohort effect but also a quadratic age effect. The two factors $A_1(t)$ and $A_2(t)$ in the CBD model represent all age general improvements in mortality over time and different improvements for different age groups. These two factors reflect both the trend effect and the age effect. Thus, the analysis will be economically or *biologically* meaningful when we consider the parameter changes of the factors over time. Second, the CBD model is a discrete-time model and can be more conveniently implemented in practice. This paper offers a brief description of the two-factor model; for a more detailed discussion, see Cairns et al. (2006b).

Let $q_{t,x}$ be the realized mortality rate for age $x$ insured from time $t$ to $t+1$. Assume that the mortality curve has a logistic functional form as follows:

$$q_{t,x} = \frac{e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}{1+e^{A_1(t+1)+A_2(t+1)\cdot(x+t)}}. \tag{1}$$

The two stochastic trends $A_1(t+1)$ and $A_2(t+1)$ follow a random-walk process with drift parameter $\mu$ and diffusion parameter $C$:

$$A(t+1) = A(t) + \mu + CZ(t+1), \tag{2}$$

where $A(t+1) = (A_1(t+1), A_2(t+1))^{\mathrm{T}}$ and $\mu = (\mu_1, \mu_2)^{\mathrm{T}}$ are $2 \times 1$ constant parameter vectors. $C$ is a $2 \times 2$ constant upper-triangular Cholesky square-root matrix of the covariance matrix $V = CC^{\mathrm{T}}$ and $Z(t)$ is a two-dimensional standard normal random variable. To include the uncertainty of $\mu$ and $C$, Cairns et al. (2006b) invoke a normal-inverse-Wishart distribution from a non-informative prior distribution:

$$\begin{aligned}
&V^{-1}\,|D \sim Wishart(n-1, n^{-1}\hat{V}^{-1})\\
&\mu^{-1}\,|V, D \sim MVN(\hat{\mu}, n^{-1}V),\\
&\text{where } D(t) = A(t) - A(t-1),\\
&\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} D(t),\\
&\text{and } \hat{V} = \frac{1}{n}\sum_{t=1}^{n}\left(D(t)-\hat{\mu}\right)\left(D(t)-\hat{\mu}\right)^{\mathrm{T}}.
\end{aligned} \tag{3}$$

Thus, we can generate $A(t)$ from Eq. (2) with the parameters $\mu$ and $C$ from Eq. (3). Then we get $q_{t,x}$, as Eq. (1) suggests.

### 2.2. The Mortality Duration Matching (MDM) method

Wang et al. (forthcoming) propose the MDM approach to calculate an optimal life insurance/annuity weight to immunize the value change from mortality risk. They propose the following product mix of life insurance:

$$w^D = \frac{D^a}{D^a + D^l}, \tag{4}$$

where $D^a$ denotes the effective duration of the annuity and $D^l$ denotes the effective duration of the life insurance. Formally, the effective duration can be calculated as follows:

$$D^a = -\frac{V^{a+} - V^{a-}}{2V^a\Delta q} \quad \text{and} \quad D^l = \frac{V^{l+} - V^{l-}}{2V^l\Delta q}.$$

The $\Delta q$ refers to the change in the mortality rate, $V^{a+}$ and $V^{l+}$ represent the product values at higher mortality rate $(q+\Delta q)$ and $V^{a-}$ and $V^{l-}$ represent the values at the lower mortality rate $(q-\Delta q)$. If the change is small, this strategy leads to the product immunization as follows:

$$\varDelta V = w^D D^l - (1 - w^D) D^a = 0. \tag{5}$$

Wang et al. (forthcoming) also propose the mortality convexity adjustment for a large change as

$$C^a = \frac{V^{a-} + V^{a+} - 2V^a}{V^a(\varDelta q)^2} \quad \text{and} \quad C^l = \frac{V^{l-} + V^{l+} - 2V^l}{V^l(\varDelta q)^2}.$$

Then the product mix weight with convexity on life insurance is

$$w^C = \frac{D^a - \frac{\Delta q}{2}C^a}{D^a + D^l + \frac{\Delta q}{2}\left(C^l - C^a\right)}. \tag{6}$$

Here, the change is set as $\Delta q = \bar{q}(1+s) - \bar{q}$, where $\bar{q}$ is the mean of the mortality process and $s$ is a shift proportion such as 1%. Thus, the change here involves a parallel shift in the mean.

[1] The non-systematic risk of products is not considered here. We assume that the non-systematic mortality risks are all diversified across policyholders via the law of large numbers. Shareholders bearing non-systematic mortality risk are not rewarded. We also assume that insurance companies will not suffer from insolvency risk.