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Biometric worst-case scenarios for multi-state life insurance policies

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ABSTRACT

It is common actuarial practice to calculate premiums and reserves under a set of biometric assumptions that represent a worst-case scenario for the insurer. The new solvency regime of the European Union (Solvency II) also uses worst-case scenarios for the calculation of solvency capital requirements for life insurance business. Surprisingly, the actuarial literature so far offers no exact method for the construction of biometric scenarios that let premiums and reserves be always on the safe side with respect to a given confidence band for the biometric second-order basis. The present paper partly fills this gap by introducing a general method that allows one to construct such scenarios for homogenous portfolios of life insurance policies. The results are especially informative for life insurance policies with mixed character (e.g. survival and occurrence character). Two examples are given that illustrate the new method, demonstrate its usefulness for the calculation of premiums and reserves, and show how the new approach could improve the calculation of biometric solvency reserves for Solvency II.

1. Introduction

By statute, a life insurer must currently maintain a reserve in order to meet all future liabilities related to its life insurance portfolio. As mortality probabilities, disability probabilities, reactivation probabilities, etc. can vary significantly within a contract period, a major concern of regulators and insurers is the choice of the biometric (or demographic) valuation basis, which is the set of biometric assumptions underlying the calculation of premiums and reserves. In practice, the true values are not perfectly known. In order to set premiums and reserves on the safe side, it is a common method to choose a first-order valuation basis that represents some worst-case scenario for the insurer.

Without any further restrictions, the mathematical worst cases for premiums and reserves are too extreme for practice or even do not exist. For example, for a temporary life insurance, the mathematical worst-case mortality rate is a scenario where the survival probability goes to zero as quickly as possible just after the beginning of the contract, or for a lifelong annuity the mathematical worst case is that the survival probability stays at 1 till the maximum age. In practice, by applying statistical methods on data of the past, we can usually narrow future uncertainties down to confidence bands for the parameters of the biometric valuation basis. For the calculation of premiums and reserves, we should now choose a first-order valuation basis that maximizes the prospective reserve with respect to all biometric scenarios within those confidence bands. It is the main contribution of this paper to introduce a method that allows one to find such a maximizing valuation basis (or worst-case scenario) for single life insurance policies. It turns out that to some extent the results are still valid on a portfolio level.

So far, the literature offers three concepts for the construction of a biometric first-order valuation basis. First, there is the method based on the sum-at-risk, which was developed by Lidstone (1905), Norberg (1985), Hoem (1988), Ramlau-Hansen (1988), and Linnemann (1993). For a given first-order basis with corresponding sums-at-risk for the different transitions, they showed that premiums and reserves are on the safe side if the second-order transition rates are smaller and greater than the first-order transition rates according as the first-order sums-at-risk are positive and negative, respectively. The problem here is that we only know that premiums and reserves are conservative with respect to those second-order transition rates that cross the first-order transition rates only where the first-order sums-at-risk are zero. Hence, in many examples the sum-at-risk method guarantees to be on the safe-side only for a small minority of possible second-order transition rates within the confidence bounds. In other words, we can not relate our first-order basis to a majority of possible second-order scenarios. This problem would vanish if we had a first-order basis that is equal to the upper and lower bounds of the confidence bands where ever the first-order sums-at-risk are positive and negative, respectively. But the sum-at-risk method

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does not tell us how to find such special first-order bases. It is a main contribution of this paper to find them, and by that we solve the maximization problem for the prospective reserve.

The second method for the construction of biometric first-order valuation bases to be found in the literature is based on derivatives. References using such an approach are Dienst (1995), Bowers et al. (1997), Christiansen and Helwich (2008), or Christiansen (2008a,b). The trick is to approximate the relevant functions with local linearizations by using first-order derivatives, since worst-case scenarios can be found much more easily for linear mappings. But by using a local concept, one can – strictly speaking – only study infinitesimal changes of the valuation basis. The method based on derivatives may yield good approximations if the confidence bands for the second-order basis are not too wide, but still the approximation error is difficult to control. In contrast, the method given in the present paper always yields exact results even if the confidence bands are arbitrarily wide.

A third method was presented by Kalashnikov and Norberg (2003). By applying the delta method, they calculated asymptotic probability distributions for premiums and reserves, given that the valuation basis parameters have an asymptotically normal distribution. This way, they obtained confidence intervals for premiums and reserves directly in one step. (In contrast, the other methods use a two-step approach. First, they construct confidence bands for the valuation basis, the input of the mappings reserve and premium. Then they study the effect that changes of the input have on the output in order to obtain confidence intervals for the reserve and premium.) The one-step approach is very appealing, but at present it runs counter to the traditional rules of insurance regulation in many countries.

The paper is organized as follows. After introducing the general modeling framework for multi-state life policies in Section 2, Section 3 presents the main contribution of this paper by formulating a maximization problem – finding a worst-case valuation basis – and giving a solution for it. In order to present the new results more fluently, two long proofs are put into an extra section, Section 4. Section 5 illustrates the new method with two examples, demonstrating the usefulness for the calculation of premiums and technical reserves and the calculation of solvency reserves in Solvency II.

2. The Markov model

Consider an insurance policy that is issued at time 0, terminates at a fixed finite time T, and is driven by a Markovian jump process $(X_t)_{t\in[0,T]}$ with finite state space $\mathcal{S}$. Let $J := \{(j,k) \in \mathcal{S}^2 \mid j \neq k\}$ denote the transition space. Following Milbrodt and Stracke (1997), we assume that $(X_t)_{t\in[0,T]}$ has a regular cumulative transition intensity matrix q. (The matrix elements q_{jk}, $j \neq k$, are nondecreasing and right-continuous functions with left-hand limits, zero at time zero, $q_{jj} = -\sum_{k\neq j} q_{jk}$, $\Delta q_{jj}(t) \geq -1$ for all t, and when $\Delta q_{jj}(t_0) = -1$ the function $q_{jj}(t)$ is constant on $[t_0, T]$.) That means that the transition probability matrix

$$p(s,t) = (P(X_t = k \mid X_s = j))_{(j,k)\in\mathcal{S}^2}, \quad 0 \leq s \leq t \leq T,$$

has a representation of the form

$$\begin{aligned} p(s,t) &= \prod_{(s,t]} (\mathbb{I} + \mathrm{d}q) \\ &:= \lim_{\max|t_i - t_{i-1}|\to 0} \prod (\mathbb{I} + q(t_i) - q(t_{i-1})), \end{aligned} \tag{2.1}$$

for partitions $s < t_0 < t_1 < \cdots < t_n = t$.

The continuous-time Markov model based on transition intensities $\mu_{jk}(t)$ and a differentiable discounting function has already been described by Sverdrup (1965) or Hoem (1969). The approach of Milbrodt and Stracke (1997) made it possible to jointly comprise the 'discrete model' and the 'continuous model' of insurance mathematics as well as intermediate cases.

Payments between the insurer and the policyholder are of two types.

(a) Lump sums are payable upon transitions $(j,k) \in J$ between two states and are specified by deterministic nonnegative functions b_{jk} with bounded variation. To distinguish between the time of transition and the actual time of payment, Milbrodt and Stracke (1997) introduced an increasing function $DT : (0,\infty) \to (0,\infty)$, $DT(t) \geq t$, such that upon transition from j to k at time t the amount $b_{jk}(t)$ is payable at time $DT(t)$. To simplify notation, assume that $DT(T) = T$.

(b) Annuity payments fall due during sojourns in a state and are defined by deterministic functions B_j, $j \in \mathcal{S}$, which for each time $t \geq 0$ specify the total amount $B_j(t)$ paid in $[0,t]$. Assume that each B_j is right-continuous and of bounded variation.

Assume that the investment portfolio of the insurance company bears interest with cumulative intensity Φ, which shall here be a nondecreasing and right-continuous function that is zero at time zero. Then the value at time s of a unit payable at time $t > s$ is

$$\begin{aligned} v(s,t) &:= \left(\prod_{(s,t]} (1 + \mathrm{d}\Phi) \right)^{-1} \\ &= \mathrm{e}^{-(\Phi^c(t) - \Phi^c(s))} \prod_{\tau\in(s,t]} \big(1 + \Delta\Phi(\tau)\big)^{-1} \end{aligned} \tag{2.2}$$

(see Milbrodt and Stracke, 1997, p. 189), where $\Phi^c(t) := \Phi(t) - \sum_{\tau\leq t} \Delta\Phi(\tau)$ for all t.

Let $V_i(s)$ be the conditional expected value of all discounted payments that fall due strictly past s given state i at time s, which is denoted as the prospective reserve at time s in state i. According to Lemma 4.4(c) and Theorem 4.8 in Milbrodt and Stracke (1997), the formula

$$\begin{aligned} V_i(s) &= \sum_{j\in\mathcal{S}} \int_{(s,T]} v(s,t)\, p_{ij}(s,t)\, \mathrm{d}B_j(t) \\ &\quad + \sum_{(j,k)\in J} \int_{(s,T]} v(s, DT(t))\, b_{jk}(t)\, p_{ij}(s,t-)\, \mathrm{d}q_{jk}(t) \end{aligned} \tag{2.3}$$

gives a version of the prospective reserve, which from now on will be used as a definition for $V_i(s)$. Further on, we have Thiele's integral equation system

$$\begin{aligned} V_i(s) &= B_i(T) - B_i(s) - \int_{(s,T]} V_i(t-)\, \mathrm{d}\Phi(t) \\ &\quad + \sum_{j:j\neq i} \int_{(s,T]} R_{ij}(t)\, \mathrm{d}q_{ij}(t) \end{aligned} \tag{2.4}$$

for all $i \in \mathcal{S}$ and $s \in [0,T]$, where $R_{ij}(t)$ is the so-called sum-at-risk associated with a possible transition from state i to state j at time t,

$$R_{ij}(t) := v(t, DT(t))\, b_{ij}(t) + V_j(t) + \Delta B_j(t) - V_i(t) - \Delta B_i(t).$$

Differing from Milbrodt and Stracke (1997), the prospective reserve is here defined in a right-continuous manner. From (2.3), we get an initial condition for Thiele's integral equation system: $V_i(T) = 0$ for all $i \in \mathcal{S}$. If Φ and the q_{jk} have the intensities φ and μ_{jk}, then (2.4) has the form

$$\begin{aligned} V_i(s) &= B_i(T) - B_i(s) - \int_{(s,T]} V_i(t-)\, \varphi(t)\, \mathrm{d}t \\ &\quad + \sum_{j:j\neq i} \int_{(s,T]} R_{ij}(t)\, \mu_{ij}(t)\, \mathrm{d}t. \end{aligned} \tag{2.5}$$