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Risk analysis and valuation of life insurance contracts: Combining actuarial and financial approaches

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ABSTRACT

In this paper, we analyze traditional (i.e. not unit-linked) participating life insurance contracts with a guaranteed interest rate and surplus participation. We consider three different surplus distribution models and an asset allocation that consists of money market, bonds with different maturities, and stocks. In this setting, we combine actuarial and financial approaches by selecting a risk minimizing asset allocation (under the real world measure **P**) and distributing terminal surplus such that the contract value (under the pricing measure **Q**) is fair. We prove that this strategy is always possible unless the insurance contracts introduce arbitrage opportunities in the market. We then analyze differences between the different surplus distribution models and investigate the impact of the selected risk measure on the risk minimizing portfolio.

1. Introduction

Interest rate guarantees are a very common product feature within traditional participating life insurance contracts in many markets. There are two major types of interest rate guarantees:

The simplest interest rate guarantee is a so-called point-to-point guarantee, i.e. a guarantee that is only relevant at maturity of the contract. The other type is called cliquet-style (or year-by-year) guarantee. This means that the policy holders have an account to which every year at least a certain guaranteed rate of return has to be credited.

Cliquet-style guarantees of course may force insurers to provide relatively high guaranteed rates of interest to accounts to which a big portion of the past years' surplus has already been credited. Adverse capital market scenarios of recent years appeared to have caused significant problems for insurers offering this type of guarantee. Therefore, the analysis of traditional life insurance contracts with cliquet-style guarantees has become a subject of increasing concern for the academic world as well as for practitioners.

There are so-called financial and actuarial approaches to handling financial guarantees within life insurance contracts. The financial approach is concerned with risk-neutral valuation and fair pricing and has been researched by various authors such as Bryis and de Varenne (1997), Grosen and Jørgensen (2000), Grosen and Jørgensen (2002) or Bauer et al. (2006). Note that the concept of risk-neutral valuation is based on the assumption of a perfect (or super-) hedging strategy, which insurance companies normally do not or cannot follow (cf. e.g. Bauer et al. (2006)). If the insurer does not or cannot invest in a portfolio that replicates the liabilities, the company remains at risk and should therefore additionally perform some risk analyses. The actuarial approach focuses on quantifying this risk with suitable risk-measures under an objective 'real-world' probability-measure, cf. e.g. Kling et al. (2007a) or Kling et al. (2007b). Such approaches also play an important role e.g. in financial strength ratings or under the new Solvency II approach. Amongst others, Gatzert and Kling (2007) investigate parameter combinations that yield fair contracts and analyze the risk imposed by fair contracts for various insurance contract models, starting with a simple generic point-to-point guarantee and afterwards analyzing more sophisticated Danish- and UK-style contracts. Kling (2007) focuses on traditional German insurance contracts where the interdependence of various parameters concerning the risk exposure of fair contracts is studied. Gatzert (2008) extends the work from Gatzert and Kling (2007) where an approach to 'risk pricing' is introduced using the 'fair value of default' to determine contracts with the same risk exposure. However, this risk measure neglects real-world scenarios and is only concerned with the (risk-neutral) value of the introduced default put option. Whilst Gatzert (2008) analyzes some real-world risk generated by

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the considered contracts, the risk exposure is not incorporated in the pricing procedure.

Barbarin and Devolder (2005) introduce a methodology that allows for combining the financial and actuarial approach. They consider a contract similar to Bryis and de Varenne's (1997) with a point-to-point guarantee and terminal surplus participation. To integrate both approaches, they use a two-step method of pricing life insurance contracts: First, they determine a guaranteed interest rate such that certain solvency requirements are satisfied, using value at risk and expected shortfall risk measures. Second, to obtain fair contracts, they use risk-neutral valuation and adjust the participation in terminal surplus accordingly.

In the present work we extend Barbarin and Devolder's (2005) methodology which then allows the pricing of life insurance contracts in a more general liability framework including in particular typical product features of the German insurance market, and an asset allocation that consists of money market, bonds with different maturities and stocks. We identify parameter combinations that minimize the real world risk without changing the fair value of the contract. We prove that the proposed methodology works unless the insurance contract design introduces arbitrage opportunities.

The remainder of this paper is organized as follows. After an introduction of the considered financial market, the insurer's asset allocation, and different liability models in Section 2, Section 3 presents our methodology of combining the actuarial and financial approach and the theoretical result that the strategy we propose is always possible unless the insurance contracts introduce arbitrage opportunities in the market. In Section 4, we show various numerical results for the introduced liability models, focusing on both, the risk a specific contract design and asset allocation imposes on the insurance company and the valuation of the contract from the client's perspective. We further investigate how the results depend on the risk measure used. Section 5 concludes.

2. Model framework

2.1. Insurance company

Following Kling et al. (2007a), we consider a simplified 'balance sheet' of the insurance company as follows:

Assets	Liabilities
	$L(t)$
$A(t)$	$B(t)$
	$R(t)$
$A(t)$	$A(t)$

Here, $A(t)$ denotes the market value of the company's assets. $L(t)$ represents the insurer's liabilities measured by the actuarial reserve for the insurance contracts. Every year $L(t)$ has to earn at least a fixed guaranteed interest rate i, thus $L(t+1) \geq L(t)(1+i)$. The insured can participate in the insurer's asset return exceeding the guaranteed rate in two ways: By regular surplus participation if in any year more than the guaranteed interest rate i is credited to the account L and by terminal surplus participation. $B(t)$ models a collective terminal surplus account, which is used to provide additional surplus participation at the maturity of a client's contract. This account may be reduced at any time in order to ensure the company's liquidity which leaves $B(t)$ to be an optional bonus payment and $B(t) \geq 0$ for all t. The residual value $R(t) = A(t)-(L(t)+B(t))$ denotes the (hidden) reserves of the life insurer.

2.2. Financial market

We now introduce the model for the financial market and the financial instruments in the insurer's asset portfolio. We allow investment in the money market, bonds and stocks. We use the Vasicek (1977) model for stochastic interest rates and a Geometric Brownian Motion (cf. Black and Scholes, 1973) for a reference stock or stock index.

We first specify our asset model under the real-world probability measure $\mathbf{P}$ and then switch to the risk-neutral measure $\mathbf{Q}$ which will be used for valuation purposes. We consider a probability space $(\Omega, F, \mathbf{F}, \mathbf{P})$ with the natural filtration $\mathbf{F} = F_t = \sigma((W_1(s), W_2(s)), s \leq t)$ generated by independent $\mathbf{P}$-Brownian Motions $W_1(t)$ and $W_2(t)$ and let $r(t)$ denote the short-rate and $S(t)$ the value of the stock at time t.

The asset model is then given by the stochastic differential equations (SDEs)

$$\mathrm{d}r(t) = a(b-r(t))\mathrm{d}t + \sigma_r \mathrm{d}W_1(t)$$

$$\mathrm{d}S(t) = S(t)(\mu \mathrm{d}t + \sigma_S(\rho \mathrm{d}W_1(t) + \sqrt{1-\rho^2}\mathrm{d}W_2(t)))$$

with correlation $\rho \in [-1, 1]$. To simplify notation, we let $W_3(t) := \rho W_1(t) + \sqrt{1-\rho^2}W_2(t)$.[3] Thus, for $t_1 \leq t_2$, a closed form solution of the above SDEs is given by[4]

$$r(t_2) = \mathrm{e}^{-a(t_2-t_1)}r(t_1) + b(1-\mathrm{e}^{-a(t_2-t_1)}) + \sigma_r \mathrm{e}^{-at_2}\int_{t_1}^{t_2} \mathrm{e}^{au}\mathrm{d}W_1(u)$$

$$S(t_2) = S(t_1)\mathrm{e}^{\left\{\left(\mu-\frac{1}{2}\sigma_S^2\right)(t_2-t_1)+\sigma_S(W_3(t_2)-W_3(t_1))\right\}}.$$

A money market investment is then modeled by an investment in the short rate: $\beta(t) = \mathrm{e}^{\int_0^t r(s)\mathrm{d}s}$.

We further consider a bond portfolio consisting of different zero-bonds. Hence we need to determine $p(t,T)$, the price at time t of a zero-bond with maturity T. We assume that $p(t,T) = F(t, r(t))$ holds for some smooth function $F(t, r(t))$. Since the short rate is not observable on the market we may not be able to hedge derivatives on the short rate (e.g. zero-bonds) by investing in the underlying itself as it could be done e.g. in a Black–Scholes framework. Investing in the bank account instead would result in an incomplete market.

By constructing a portfolio with no instantaneous risk (e.g. consisting of two zero-bonds with different maturities) and applying no arbitrage arguments, one arrives at the so-called *market price of risk* $\lambda(t, r(t))$ and hence at a partial differential equation for zero-bond prices,[5] the so-called *term structure equation*.

$$F_t(t, r(t)) + (a(b-r(t)) - \lambda(t, r(t))\sigma_r)F_r(t, r(t)) + \frac{1}{2}\sigma_r^2 F_{rr}(t, r(t)) - r(t)F(t, r(t)) = 0$$

with terminal condition $F(T, r(T)) = 1$.

The Feynman–Kaç[6] formula then allows for a probabilistic interpretation of the above partial differential equation by

$$p(t,T) = F(t, r(t)) = \mathbf{E}_{\tilde{\mathbf{Q}}}\left(\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}|r(t)\right)$$

with a probability measure $\tilde{\mathbf{Q}}$ and a stochastic process $r(t)$ with $\tilde{\mathbf{Q}}$-dynamics $\mathrm{d}r(t) = (a(b-r(t)) - \lambda(t, r(t))\sigma_r)\mathrm{d}t + \sigma_r \mathrm{d}\tilde{W}_1$. Note that observed zero-bond prices induce the market price of risk $\lambda(t, r(t))$ and therefore no obvious form or parameterization of $\lambda(t, r(t))$ exists ad hoc. However, if and only if we assume $\lambda(t, r(t)) = \lambda$, the short rate process under $\tilde{\mathbf{Q}}$ remains of the Vasiçek-type. From standard interest rate theory (cf. e.g. Björk, 2005) it follows that $p(t,T) = \mathrm{e}^{A(t,T)-B(t,T)r(t)}$ with $A(t,T) =$

[3] From Lévy's theorem it follows that $W_3(t)$ is a $\mathbf{P}$-Brownian Motion as well.

[4] Compare Björk (2005) for further details.

[5] For a complete derivation cf. Graf (2008).

[6] Cf. e.g., Björk (2005).