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The uncertain mortality intensity framework: Pricing and hedging unit-linked life insurance contracts

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ABSTRACT

We study the valuation and hedging of unit-linked life insurance contracts in a setting where mortality intensity is governed by a stochastic process. We focus on model risk arising from different specifications for the mortality intensity. To do so we assume that the mortality intensity is almost surely bounded under the statistical measure. Further, we restrict the equivalent martingale measures and apply the same bounds to the mortality intensity under these measures. For this setting we derive upper and lower price bounds for unit-linked life insurance contracts using stochastic control techniques. We also show that the induced hedging strategies indeed produce a dynamic superhedge and subhedge under the statistical measure in the limit when the number of contracts increases. This justifies the bounds for the mortality intensity under the pricing measures. We provide numerical examples investigating fixed-term, endowment insurance contracts and their combinations including various guarantee features. The pricing partial differential equation for the upper and lower price bounds is solved by finite difference methods. For our contracts and choice of parameters the pricing and hedging is fairly robust with respect to misspecification of the mortality intensity. The model risk resulting from the uncertain mortality intensity is of minor importance.

1. Introduction

Mortality is a major risk factor for life insurance companies and pension funds that needs to be modeled properly. In recent years, it has been widely accepted that mortality changes over time in an unpredictable way and stochastic models have been developed to adequately capture the systematic mortality risk. For stochastic models valuing of mortality-linked liabilities and determining the required market reserves, see for instance Milevsky and Promislow (2001), Dahl (2004), Biffis (2005), Dahl and Møller (2006), and Young (2008). Stochastic models with an emphasis on securitizing mortality risk by introducing survivor bonds as hedging instruments are discussed by e.g., Blake et al. (2006) and Cairns et al. (2006). Each mortality model is a possible description of the mortality risk. Melnikov and Romaniuk (2006) show that different mortality models perform differently in the risk management of a unit-linked pure endowment contract and warn us to be careful when choosing one mortality model against another. In this paper we provide a framework for assessing the mortality model risk embedded in unit-linked life insurance contracts arising from different specifications for the mortality intensity.

Unit-linked life insurance contracts are popular and widely used on the insurance market.[1] They provide either death benefit or maturity benefit or both. The benefits are linked to an underlying asset with or without certain guarantees so that the policyholders have the opportunity to participate in the financial market and (eventually) be protected from the downside development of the financial market. Many unit-linked life insurance contracts also embed options in them, e.g., the surrender option allowing the policyholders to terminate the contracts prematurely and the guaranteed annuity option giving the policyholders the right to convert a lump sum payment at the maturity into annuities at a predetermined rate. Depending on the payoff structures of the contracts, the effect of the mortality model risk may also be different. By investigating the effect of the mortality model risk

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[1] In the 1990s, unit-linked life insurance was very popular in household financial planning. The share of unit-linked premiums increased from 20% in 1997 to 36% in 2001 of the total life insurance premiums which accounted for over 10% of the GDP of western Europe; see Swiss Re (2003). Although the enthusiasm in unit-linked insurance from the policyholder side declined during financial market crashes, e.g., at the end of 2001 and between 2007 and 2010, this business is expected to boom again when the capital market recovers from the depression. According to Swiss Re (2003), a simple regression analysis shows that a 10% rise in the stock market led to a 15% increase in single-premium unit-linked sales.

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we are able to know whether its importance is under or over-emphasized for different contract types.

In our paper instead of inputting different mortality models into the same pricing and hedging problem and comparing their performances as Melnikov and Romaniuk (2006), we set up a more flexible framework saying that we do not know the exact process of the mortality intensity but are able to figure out its upper and lower bound under the statistical measure. Further, we restrict the set of equivalent martingale measures such that the same bounds apply to the mortality intensity under these measures. This setup allows us to study various contract types more efficiently and we call it the uncertain mortality intensity framework; see Avellaneda et al. (1995) for a related framework for pricing stock options when the volatility process is unknown but bounded.

Within our framework we do not intend to find the fair value of a contract but its price bounds. The price bounds are solutions to the partial differential equations associated to a stochastic control problem. The upper price bound is found by choosing the worst-case mortality intensity at any time during the life time of the contract so that the contract value is maximized. Whereas the lower price bound is found by setting the mortality intensity to the best-case value in the sense that the contract value would be minimized. The effect of our approach is quite similar to that of the practice in traditional life insurance like pure endowment insurance and term insurance. An insurance company usually puts itself on the safe side by adjusting the premium by a loading factor defined as a percentage markup from the actuarially fair value of insurance. This is equivalent to assuming lower mortality intensity for pure endowment insurance and higher mortality intensity for term insurance. However, since our approach chooses the worst (or best) possible mortality intensity dynamically, we are able to deal with more complex contract structures where the safest mortality intensity at any time also depends on the price of the underlying asset. As a result, the higher the difference between the upper and the lower price bounds, the greater impact would the mortality model risk have on the contracts considered. In this way we are able to identify whether model risk is potentially deteriorating the fair evaluation of the contracts.

Further we examine hedging strategies induced by the price bounds. The unsystematic mortality risk is diversified by pooling a large enough number of policyholders together as usually is the case. However, the systematic mortality risk, that is here the random fluctuations of the mortality intensity, can in general not be diversified away by using the pooling rationale. Instead of applying risk-minimizing or mean–variance hedging strategies to minimize either hedging costs or hedging error (see Dahl and Møller (2006) and Young (2008)) we suggest using hedging strategies induced by the upper and lower price bounds. By construction, these strategies produce a superhedge and subhedge, respectively, on average for an increasing number of policyholders. We provide numerical examples investigating fixed-term, endowment insurance contracts and their combinations including various guarantee features. The pricing partial differential equation for the upper and lower price bounds is solved by finite difference methods. For our contracts and choice of parameters pricing and hedging is fairly robust with respect to misspecification of the mortality intensity, with at most a mispricing of 4% for single premium contracts and at most 2% for periodic premium payment. We conclude that model risk resulting from the uncertain mortality intensity is of minor importance.

The structure of the paper is as follows. In Section 2 we describe both the financial market and the insurance market. In Section 3 we formalize the uncertain mortality intensity framework. Based on the model setup, we introduce in Section 4 the optimal control rule of the mortality intensity within its upper and lower bounds so that the price bounds are found. This enables us to build in mean superhedging strategies which are discussed in Section 5. Section 6 illustrates the theoretical results by providing a numerical analysis for different types of unit-linked life insurance contracts. Section 7 concludes.

2. Setup

The model for the financial market and the insurance market is developed subsequently. Both markets are jointly specified on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$. The probability $\mathbb{P}$ is called the real world measure and is sometimes also referred to as statistical measure. We assume that the probability space is large enough to support an n-dimensional Wiener process $W = [W^1, W^2, \dots, W^n]$ and a random time τ. The time horizon is denoted by T.

The financial market consists of a risky asset with price process S and a riskless money market account with price process B. The latter is given by $B_t = \exp\left\{\int_0^t r(u)\,\mathrm{d}u\right\}, 0 \le t \le T$, where the risk-free interest rate r is a deterministic and continuous function. The risky asset price process S is governed by the stochastic differential equation

$$\mathrm{d}S_t = \alpha(t, S_t)\, S_t\, \mathrm{d}t + \sigma(t, S_t)\, S_t\, \mathrm{d}W_t^1, \quad 0 \le t \le T, \tag{1}$$

where α is the local mean rate of return and σ is the volatility. The dividend structure D is given by

$$\mathrm{d}D_t = q(t, S_t) S_t \mathrm{d}t, \quad 0 \le t \le T, \tag{2}$$

where q is a continuous deterministic function.[2] The financial market modeled in this way is complete and arbitrage free and is called $\mathbb{F}^S$ market. Here, $\mathbb{F}^S = \left(\mathcal{F}_t^S\right)_{0\le t\le T}$ is the augmented natural filtration generated by the stock price process S. Since $\sigma > 0$ it follows that the augmented natural filtration generated by the first component W^1 of the Wiener process $\mathbb{F}^1 = \left(\mathcal{F}_t^{W^1}\right)_{0\le t\le T}$ coincides with the market filtration $\mathbb{F}^S$.

The insurance market is modeled by the random time τ denoting the death time of an individual aged x at the starting time 0.[3] For simplicity of notation we will omit the age variable x in the subsequent discussion of mortality related variables. The filtration generated by the right-continuous indicator process $H_t = 1_{\{\tau\le t\}}$, for $t \in [0, T]$, is denoted $\mathbb{H} = (\mathcal{H}_t)_{0\le t\le T}$. The mortality is potentially influenced by an m-dimensional environment process $X = [X_1, \dots, X_m]$ with dynamics

$$\mathrm{d}X_t = \alpha_X(t, X_t)\, \mathrm{d}t + \Sigma_X(t, X_t)\, \mathrm{d}W_t, \quad 0 \le t \le T, \tag{3}$$

where α_X is a $\mathbb{R}^m$-valued function and Σ_X is a $\mathbb{R}^{n\times m}$-valued function, both regular enough to ensure the existence of a solution to the SDE. By definition it is clear that X is adapted to the filtration generated by the Wiener process W, say, $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$. Note that $\mathbb{F}^S \subseteq \mathbb{F}$, and further denote the joint filtration by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. The financial market model for unit-linked life insurance contracts is then called the $\mathbb{G}$ market.

2.1. Dependence of financial market and insurance market

The probabilistic connection between W and τ is now formalized. In broad terms we assume that we are in a setting frequently used in the credit risk literature; see Bielecki and Rutkowski (2001), part II, for a detailed treatment. In particular, we assume that on $(\Omega, \mathcal{G}, \mathbb{P})$ there exists a unit exponentially distributed random variable E_1 that is independent of W and further that there

[2] We assume that the coefficients α and σ are regular enough to ensure the existence of a solution to the SDE (1); see for instance Protter (2004), Chapter V, Section 3. Additionally, we assume that α, σ and q are uniformly bounded and σ is bounded away from zero to ensure the integrability of S, related portfolio value processes, and to ensure the existence of the measure change from $\mathbb{P}$ to an equivalent martingale measure $\mathbb{Q}$.

[3] In Section 5 we consider the case of a family of random times $(\tau_i)_{i\ge1}$ and the corresponding contracts.