Insurance: Mathematics and Economics 50 (2012) 1–11

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Discrete-time local risk minimization of payment processes and applications to equity-linked life-insurance contracts

Jérôme Pansera *

Department of Statistics and Actuarial Science, University of Iowa, Iowa City, IA 52242, USA

ARTICLE INFO

Article history:
Received October 2010
Received in revised form September 2011
Accepted 4 October 2011

Keywords:
Local risk minimization
Locally risk-minimizing hedging strategy
Minimal martingale measure
Incomplete market
Enlargement of filtration
Equity-indexed annuity
Unit-linked life insurance

ABSTRACT

We develop a theory of local risk minimization for payment processes in discrete time, and apply this theory to the pricing and hedging of equity-linked life-insurance contracts. Thus, we extend the work of Møller (2001a) in several directions: from risk minimization (which is done under a martingale measure) to local risk minimization (which is done under an arbitrary measure), from single claims to payment processes, from complete financial markets to possibly incomplete financial markets, from a single risky asset to several risky assets, and from finite state spaces to general state spaces.

Moreover, we show that, when tradable financial assets are independent of mortality, a locally risk-minimizing hedging strategy for most claims in the combined financial and mortality market (such as those arising from equity-indexed annuities) may be expressed as the product of two simpler locally risk-minimizing hedging strategies: one for a purely financial claim, the other for a traditional (i.e. non-equity-linked) life-insurance claim.

Finally, we also show, under general assumptions, that the minimal measure for the combined market is the product of the minimal measure for the financial market and the physical measure for the mortality.

1. Introduction

The aim of this paper is to study local risk minimization of payment processes in discrete time, particularly in the context of equity-linked life-insurance contracts. For such contracts, the typical (discounted) monthly payment consists of a sum of terms of the form

$$H^S \cdot H^T, \qquad (1)$$

where the random variable H^S is contingent on the price history of stocks, mutual funds, options, and bonds, while the random variable H^T is contingent on the survival or death of a policyholder. For example, H^S might be the payoff of an S&P500 option and H^T, the indicator function of the event "the policyholder is still alive". More generally, H^S can be interpreted as a *purely financial claim*, and H^T, as a *traditional* (i.e. non-equity-linked) *life-insurance claim*.

While the no-arbitrage price of a purely financial claim is based on the expected value of its discounted payoff under a risk-neutral probability measure Q, the actuarial price of a traditional life-insurance claim is based – according to the equivalence principle – on the expected value of its discounted payoff under the real-world probability measure P. Hence, when pricing the *product claim* (1), which is a mixture of financial and insurance claims, it is not immediately obvious which methodology to adopt.

An elegant solution to this problem is provided by *local risk minimization*, a general approach to price and hedge claims, introduced by Schweizer (1988). Roughly speaking, the idea is to minimize, at every trading period, the mean square hedging error (i.e. the cost increment) of a not necessarily self-financing hedging strategy for a given claim. The initial value of this *locally risk-minimizing strategy* can then be interpreted as a *fair price* for the claim.

Remarkably, this fair price agrees not only with the no-arbitrage price of a financial claim (in the case of a complete market), but also with the actuarial price of an insurance claim. It is then only natural to use local risk minimization to price and hedge the product claim (1). This idea was pioneered by Møller (1998, 2001a), who considers *risk minimization* (a particular case of local risk minimization) of equity-linked life-insurance contracts in the context of a complete (Black–Scholes or Cox–Ross–Rubinstein) financial market.

Local risk minimization leads to convenient linear pricing and hedging rules: the fair price and the locally risk-minimizing strategy of a sum of claims $H_1 + \cdots + H_n$ are the sum of the fair prices $\pi_1 + \cdots + \pi_n$ and the sum of the locally risk-minimizing strategies $\varphi_1 + \cdots + \varphi_n$, respectively, of these claims. This linearity property simplifies the pricing and hedging of a *payment process* (i.e. a series of claims occurring at different times) and of a portfolio of several equity-linked life-insurance contracts.

* Tel.: +1 319 335 0824; fax: +1 319 335 3017.
E-mail addresses: jerome-pansera@uiowa.edu, jpansera@stat.uiowa.edu.

0167-6687/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2011.10.001

Local risk minimization also leads to an intuitive factorization property: when the financial claim H^S and the insurance claim H^T are independent, the fair price and the locally risk-minimizing strategy of the product claim (1) are the product of the fair prices $\pi^S\pi^T$ and the product of the locally risk-minimizing strategies $\varphi^S\varphi^T$, respectively, of the two claims H^S and H^T. This factorization property is observed by Møller (1998, 2001a), for risk minimization, and by Barbarin (2007a,b, 2008a,b), for continuous-time (local) risk minimization. Here, we establish this factorization property (Theorem 1 and Corollary 2) for discrete-time local risk minimization under a general setting: we consider multi-dimensional asset price processes taking values in a general state space, incomplete financial markets, path-dependent contingent claims, and payment processes.

Moreover, the fair price of a claim can be written as the expected discounted value of its payoff under a so-called *minimal measure*. We show (Theorem 2) that when the probability space $\left(\Omega^S, \mathbb{F}^S, \mathrm{P}^S\right)$ governing the financial market is independent of the probability space $\left(\Omega^T, \mathbb{F}^T, \mathrm{P}^T\right)$ governing mortality, the minimal measure on the product of these two spaces is the product of the minimal measure $\hat{\mathrm{Q}}^S$ for the financial market and the real-world measure P^T for mortality. This result makes it possible to price, in the combined financial and mortality market, claims that cannot necessarily be written as the product of a financial claim H^S and an insurance claim H^T.

This paper is organized as follows. In the next section, we review the literature on (local) risk minimization and its applications to life insurance. Section 3 develops a theory of local risk minimization for payment processes in discrete time. In that section, local risk minimization is considered in a general framework, without any particular application to life insurance. Section 4 introduces mortality in our framework, and applies local risk minimization to equity-linked life-insurance contracts. Section 5 concludes the paper.

2. Literature overview

The idea of minimizing the mean square cost of a (non-self-financing) hedging strategy under a martingale measure, i.e. *risk minimization*, goes back to Föllmer and Sondermann (1986). This idea is then extended to the case of an arbitrary measure, i.e. *local risk minimization*, by Schweizer (1988). The concept of local risk minimization, along with the closely related concepts of minimal martingale measure, Föllmer–Schweizer decomposition, and Galtchouk–Kunita–Watanabe decomposition, is further clarified in Föllmer and Schweizer (1991), Schweizer (1991, 1995), Biagini and Pratelli (1999), Heath et al. (2001), and Choulli et al. (2010).

The above works deal mostly with (local) risk minimization in continuous time, while Föllmer and Schweizer (1988), Schäl (1994), and Lamberton et al. (1998) concentrate on local risk minimization in discrete time. The last two papers provide several definitions, along with criteria for their equivalence. See also the recent paper of Cerny and Kallsen (2009).

A summary of various quadratic hedging techniques (risk minimization, local risk minimization, and others), in both continuous and discrete time, is provided by Schweizer (2001). Over the following decade, continuous-time local risk minimization was then extended to the multi-dimensional case and to payment processes by several authors independently (Barbarin, 2007a, 2008a,b; Riesner, 2007; Schweizer, 2008; Vandaele and Vanmaele, 2008b). An excellent up to date overview of (local) risk minimization is given by Vandaele (2010, Chapter 4).

As for applications of quadratic hedging techniques to life insurance, Møller has pioneered the approach in a series of papers. Risk minimization of equity-linked life-insurance contracts is done in continuous time (with a Black–Scholes financial market) in Møller (1998) and in discrete time (with a Cox–Ross–Rubinstein financial market) in Møller (2001a). Møller (2001b) also extends continuous-time risk-minimization to payment processes (again with a Black–Scholes financial market). In all cases, the financial market is complete.

In the same vein as Møller's work with risk minimization, several authors apply continuous-time (local) risk minimization to life insurance in many different contexts: stochastic interest rates (Lin and Tan, 2003), Lévy-driven (incomplete) financial markets (Riesner, 2006; Vandaele and Vanmaele, 2008a), systematic mortality risk, multi-dimensional assets, and possibly incomplete financial markets (Barbarin, 2007b, 2008b), surrender options (Barbarin, 2007a, 2008b; Vandaele and Vanmaele, 2009), and payment processes (Barbarin, 2008a,b).

Equity-linked life-insurance contracts may also be regarded as claims in a defaultable market, where "time of default" is "time of death", and local risk minimization (along with other quadratic hedging methods) is applied to defaultable claims in Biagini and Cretarola (2007). One important difference between equity-linked life-insurance and defaultable claims though, is that it is common to assume independence between tradable assets and mortality in the former, while it is more natural to suppose that tradable assets and time of default are dependent in the latter. For a study of this second situation, in a very general setting, we refer to Biagini and Cretarola (2009).

Thus, with the exception of some computational work by Coleman et al. (2006, 2007a,b), the literature on local risk minimization of life-insurance contracts, while being vast, deals mostly with the continuous-time case. Even though this continuous-time setting is arguably mathematically more elegant, the discrete-time case is closer to reality and, being much less technical, can offer valuable insight.

3. General framework

We take for granted a probability space $(\Omega, \mathbb{F}, \mathrm{P})$ endowed with a discrete-time filtration $\mathcal{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_n)$, where $\mathcal{F}_i$ represents the information available at time t_i. For convenience, we assume that $\mathbb{F} = \mathcal{F}_n$. All random variables and stochastic processes will be defined on this filtered probability space.

3.1. Tradable assets

Assume we have, at our disposal, $d + 1$ tradable assets. For example, each one of these assets may be a bond, a stock, a mutual fund, or an option, etc. Let $\tilde{S}_i^{(k)}$ be the time-t_i price of asset k, with $i \in \{0, \ldots, n\}$ and $k \in \{0, \ldots, d\}$. Thus, the $(d + 1)$-dimensional random vector $\tilde{\boldsymbol{S}}_i := \left(\tilde{S}_i^{(0)}, \ldots, \tilde{S}_i^{(d)}\right)$ denotes the vector of the time-t_i price of all tradable assets and the $(d + 1)$-dimensional process $\tilde{\boldsymbol{S}} := \left(\tilde{\boldsymbol{S}}_0, \ldots, \tilde{\boldsymbol{S}}_n\right)$, the price process of all tradable assets.

We assume this price process $\tilde{\boldsymbol{S}}$ is adapted to the filtration $\mathcal{F}$. This filtration may be simply the one generated by the process $\tilde{\boldsymbol{S}}$, but will typically contain additional information; for example, information about non-tradable assets, whether a person is alive, etc.

Moreover, we require the process $\tilde{S}^{(0)} := \left(\tilde{S}_0^{(0)}, \ldots, \tilde{S}_n^{(0)}\right)$ to be strictly positive (since this asset will be used as the numeraire), but make no such requirement for the price processes of the other assets (which may even be negative).

3.2. Discounted assets

Let

$$S_i^{(k)} := \frac{\tilde{S}_i^{(k)}}{\tilde{S}_i^{(0)}}$$