Insurance: Mathematics and Economics 51 (2012) 442–448

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

# An operator splitting harmonic differential quadrature approach to solve Young's model for life insurance risk

Luca Vincenzo Ballestra [a], Massimiliano Ottaviani [b], Graziella Pacelli [b,*]

[a] Dipartimento di Strategie Aziendali e Metodologie Quantitative, Seconda Università di Napoli, Corso Gran Priorato di Malta, 81043 Capua, Italy

[b] Dipartimento di Management, Università Politecnica delle Marche, Piazza Martelli 8, 60121 Ancona, Italy

ARTICLE INFO

*Article history:*
Received May 2012
Received in revised form
June 2012
Accepted 30 June 2012

*Keywords:*
Young's model
Life insurance
Harmonic differential quadrature

ABSTRACT

This paper is concerned with the numerical approximation of a mathematical model for life insurance risk that has been presented quite recently by Young (2007, 2008). In particular, such a model, which consists of a system of several non-linear partial differential equations, is solved using a new numerical method that combines an operator splitting procedure with the differential quadrature (DQ) finite difference scheme. This approach allows one to reduce the partial differential problems to systems of linear equations of very small dimension, so that pricing portfolios of many life insurances becomes a relatively easily task. Numerical experiments are presented showing that the method proposed is very accurate and fast. In addition, the limit behavior of portfolios of life insurances as the number of contracts tends to infinity is investigated. This analysis reveals that the prices of portfolios comprising more than five thousand policies can be accurately approximated by solving a linear partial differential equation derived in Young (2007, 2008).



## 1. Introduction

We propose a numerical method for pricing portfolios of life insurances based on a mathematical model that has been presented quite recently by Young (2007, 2008). This model is interesting for several reasons. First of all, the complex stochastic evolution of the mortality rate is taken into account; second, insurers are treated as rational agents, who minimize the risk of their portfolios; finally, the mortality risk premium required by insurance companies is suitably modeled. Note that Young's model is actually an extension to life insurance risk of a framework for pricing pure endowments that has been previously developed by Milevsky et al. (2005). Furthermore, models similar to Young's model have also been proposed for pricing portfolios of life insurances and pure endowments (Bayraktar and Young, 2007), and for pricing life annuities (Bayraktar et al., 2009).

In Young (2007, 2008) Young considers a portfolio of homogeneous life insurances (i.e., life insurances with same maturity and same face value), and finds that the price of such a portfolio satisfies a system of non-linear partial differential equations of parabolic type. Precisely, we have as many partial differential equations as is the number of policies considered. In addition, in Young (2007, 2008) it is also proven that, as the number of policies considered tends to infinity, the average price of the life insurances tends to a function that satisfies a linear partial differential equation of parabolic type. Neither the non-linear system of partial differential equations nor the linear partial differential equation have exact closed-form solutions, so they must be approached by numerical techniques. Nevertheless, to the best of our knowledge, no numerical methods to solve the system of non-linear partial differential equations have yet been proposed. Instead, in Young (2008), a finite difference scheme is employed for approximating the linear partial differential equation that holds in the limit as the number of life insurances tends to infinity. Now, the solution of such a limit equation can be used as an estimation of the true portfolio price if the number of policies considered is sufficiently large. However, it would also be interesting to solve the system of non-linear partial differential equations that holds in the finite case, at least for two reasons.

First of all, there are practical applications in which the number of life insurances to be priced is not extremely large (say smaller than some thousands). To figure this out, let us think, for instance, to the case of an insurer who wants to sell a basket of new contracts, for example to the employees of a firm; in addition, we shall also consider that Young's model is to be applied not to all the policies traded by an insurer, but only to policies that can be reasonably assumed to be homogeneous (for maturity and face value).

Second, solving the non-linear system of partial differential equations is useful to investigate how well the average life insurance price is approximated by the solution of the linear partial

* Corresponding author. Tel.: +39 071 2207050; fax: +39 071 2207150.
*E-mail address:* g.pacelli@univpm.it (G. Pacelli).

0167-6687/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2012.06.012

differential equation. In this respect it should be observed that in Young (2007, 2008) Young does not give any quantitative measure of the distance between the solution of the linear limit equation and the solution of the system of non-linear partial differential equations.

In the present paper, the system of non-linear partial differential equations arising in Young's model is solved using a new method based on the harmonic differential quadrature (HDQ) finite difference scheme. The HDQ approach, originally introduced by Striz et al. (1995), and further developed and analyzed by Chen et al. (1996) and Shu and Xue (1997), has been applied to various problems in science and engineering (see, for example, Janghorban (2011), Malekzadeh and Karami (2005), Shu (2000), Shu and Richard (1992) and Striz et al. (1997)). Its main advantage is that very accurate approximations are achieved using a computational mesh with a very small number of nodes.

In the present manuscript, the HDQ method is employed in conjunction with a suitable operator splitting technique, which allows us to decouple the non-linear partial differential problem characteristic of Young's model into independent smaller problems. By combining the HDQ method with such operator splitting procedure, we obtain systems of linear (algebraic) equations of very small dimension, so that portfolio prices can be calculated in reasonable times also when the number of life insurances is relatively large (say equal to some thousands).

Numerical experiments are presented showing that the method proposed is significantly accurate and fast. In fact, for example, portfolios of five thousand policies can be priced with an error of order $10^{-3}$ in a time equal to 69.2 s (on a computer with a Pentium P6000 1.87 GHz 4 GB RAM).

Furthermore, in this paper the convergence behavior of the portfolio price as the number of policies increases is investigated. In particular, it is shown that when there are five thousand life insurances the difference between the solution of the linear partial differential equation and the true average life insurance price starts to become of order $10^{-3}$. Thus, the prices of portfolios that comprise at least five thousand policies can be accurately approximated by solving the linear partial differential equation, in place of the system of (many) non-linear partial differential equations.

Finally, we point out that the numerical method proposed in the present manuscript could also be applied to several other models in actuarial mathematics, which, from the analytical standpoint, are substantially analogous to Young's model. Some of these models are briefly described in Appendix.

The remainder of the paper is organized as follows: in Section 2 the basic facts about Young's model are briefly recalled; in Section 3 the HDQ-operator splitting finite difference approach is developed; in Section 4 the numerical results obtained are presented and discussed; in Section 5 some conclusions are drawn.

## 2. The mathematical model

According to Young's model (Young, 2007, 2008), the mortality rate $\lambda$, which measures the probability of an individual dying in an infinitesimal time, is described by the following stochastic differential equation:

$$d\lambda = \mu(\lambda, t)(\lambda - \underline{\lambda})dt + \sigma(t)(\lambda - \underline{\lambda})dW^{\lambda}, \tag{1}$$

where $\mu$ and $\sigma$ are suitable functions of their arguments (they will be specified later), $\underline{\lambda}$ is a positive constant parameter, and $W^{\lambda}$ is a Wiener standard process.

Furthermore, in Young (2008) the interest rate $r$ is modeled according to the following stochastic differential equation:

$$dr = b(r, t)dt + d(r, t)dW^{r}, \tag{2}$$

where $b$ and $d$ are suitable functions of their arguments (they will be specified later), and $W^r$ is a Wiener standard process uncorrelated with $W^{\lambda}$. Note that in Young (2007, 2008) the vector variable $(r, \lambda)$ is assumed to vary in the following set:

$$\overline{\Omega} = [0, +\infty) \times [\underline{\lambda}, +\infty). \tag{3}$$

Let $\Omega$ denote the interior set of $\overline{\Omega}$, that is $\Omega = (0, +\infty) \times (\underline{\lambda}, +\infty)$. Moreover, let $A^{(N)}(r, \lambda, t)$ denote the price of a portfolio of $N$ homogeneous life insurances with face value 1 and maturity $T$. As shown in Young (2007, 2008), the function $A^{(N)}$ must satisfy the following system of non-linear partial differential equations:

$$\begin{aligned}
&\frac{\partial A^{(n)}}{\partial t} + b(r,t)\frac{\partial A^{(n)}}{\partial r} + \frac{1}{2}d^2(r,t)\frac{\partial^2 A^{(n)}}{\partial r^2} \\
&\quad + \mu(\lambda,t)(\lambda - \underline{\lambda})\frac{\partial A^{(n)}}{\partial \lambda} + \frac{1}{2}\sigma^2(t)(\lambda-\underline{\lambda})^2\frac{\partial^2 A^{(n)}}{\partial \lambda^2} - rA^{(n)} \\
&\quad - n\lambda(A^{(n)} - A^{(n-1)} - 1) \\
&\quad = -\alpha\sqrt{\sigma^2(t)(\lambda-\underline{\lambda})^2\left(\frac{\partial A^{(n)}}{\partial\lambda}\right)^2 + n\lambda\left(A^{(n)} - A^{(n-1)} - 1\right)^2}, \\
&\quad (r,\lambda) \in \Omega,\ t < T,\ n = 1, 2, \ldots, N,
\end{aligned} \tag{4}$$

where $\alpha$ is the so-called Sharpe ratio, which can be thought of as a mortality risk premium (see Young (2008)).

The system of partial differential equations (4) must be solved recursively for $n = 1, 2, \ldots, N$, and the function $A^{(0)}$ needed to obtain $A^{(1)}$ is given by

$$A^{(0)}(r, \lambda, t) = 0, \quad (r, \lambda) \in \overline{\Omega},\ t \le T. \tag{5}$$

The differential equations (4) must be equipped with final condition:

$$A^{(n)}(r, \lambda, T) = 0, \quad (r, \lambda) \in \overline{\Omega},\ n = 1, 2, \ldots, N. \tag{6}$$

Let $p^{(N)}$ denote the average price of the $N$ life insurances:

$$p^{(N)} = \frac{A^{(N)}}{N}. \tag{7}$$

In Young (2007, 2008) it is also shown that as $N$ tends to infinity $p^{(N)}$ tends to a function $p$ that satisfies the following linear partial differential equation:

$$\begin{aligned}
&\frac{\partial p}{\partial t} + b(r,t)\frac{\partial p}{\partial r} + \frac{1}{2}d^2(r,t)\frac{\partial^2 p}{\partial r^2} \\
&\quad + (\mu(\lambda,t) - \alpha\sigma(t))(\lambda-\underline{\lambda})\frac{\partial p}{\partial\lambda} + \frac{1}{2}\sigma^2(t)(\lambda-\underline{\lambda})^2\frac{\partial^2 p}{\partial\lambda^2} \\
&\quad - rp - \lambda(p-1) = 0, \quad (r,\lambda) \in \Omega,\ t < T,
\end{aligned} \tag{8}$$

with the final condition:

$$p(r, \lambda, T) = 0, \quad (r, \lambda) \in \overline{\Omega}. \tag{9}$$

In Young (2007, 2008) the problem of which boundary conditions to apply to (4)–(6) and to (8)–(9) is not addressed. Therefore, given that, for economic reasons, the function $A^{(n)}$ is monotone in the $\lambda$ and the $r$ variables (see also Fig. 1), and is bounded from below and above (being positive and smaller than the total face value of $n$ life insurances), we prescribe:

$$\lim_{r\to+\infty}\frac{\partial A^{(n)}(r,\lambda,t)}{\partial r} = 0, \quad \lambda \in [\underline{\lambda}, +\infty),\ t < T,\ n = 1, 2, \ldots, N, \tag{10}$$

$$\lim_{\lambda\to+\infty}\frac{\partial A^{(n)}(r,\lambda,t)}{\partial \lambda} = 0, \quad r \in [0, +\infty),\ t < T,\ n = 1, 2, \ldots, N. \tag{11}$$