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Optimal investment, consumption and life insurance under mean-reverting returns: The complete market solution[☆]

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ABSTRACT

This paper considers the problem of optimal investment, consumption and life insurance acquisition for a wage earner who has CRRA (constant relative risk aversion) preferences. The market model is complete, continuous, the uncertainty is driven by Brownian motion and the stock price has mean reverting drift. The problem is solved by dynamic programming approach and the HJB equation is shown to have closed form solution. Numerical experiments explore the impact market price of risk has on the optimal strategies.

1. Introduction

The goal of this paper is to provide optimal investment, consumption and life insurance acquisition strategies for a wage earner who uses an expected utility criterion with CRRA type preferences. Existing works on optimal investment, consumption and life insurance acquisition use a financial model in which risky assets are modeled as geometric Brownian motions. In order to make the model more realistic, we allowed the risky assets to have stochastic drift parameters. Let us review the literature on optimal investment, consumption and life insurance.

The investment/consumption problem in a stochastic context was considered by Merton (1969, 1971). His model consists in a risk-free asset with constant rate of return and one or more stocks, the prices of which are driven by geometric Brownian motions. The horizon T is prescribed, the portfolio is self-financing, and the investor seeks to maximize the expected utility of intertemporal consumption plus the final wealth. Merton provided a closed form solution when the utilities are of constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA) type. It turns out that for (CRRA) utilities the optimal fraction of wealth invested in the risky asset is constant through time. Moreover for the case of (CARA) utilities, the optimal strategy is linear in wealth.

Richard (1975) added life insurance to the investor's portfolio by assuming an arbitrary but known distribution of death time. In the same vein Pliska and Ye (2007) studied optimal life insurance and consumption for an income earner whose lifetime is random and unbounded. Ye (2008) extended Pliska and Ye (2007) by considering investments into a financial market. Huang and Milevsky (2007) solved a portfolio choice problem that includes mortality–contingent claims and labor income under general hyperbolic absolute risk aversion (HARA) preferences focusing on shocks to human capital and financial capital. Horneff and Maurer (2009) considered the problem in which an investor has to decide among short and long positions in mortality–contingent claims. Their analysis revealed when the wage earners demand for life insurance switches to the demand for annuities. Duarte et al. (2011) extended Ye (2008) to allow for multiple risky assets modeled as geometric Brownian motions. More recently Kwak et al. (2009) looked at the problem of finding optimal investment, consumption and life insurance acquisition for a family whose parents receive deterministic labor income until some deterministic time horizon.

The novelty of our work is that we allow for stochastic drift. More precisely the stock price is assumed to have mean reverting drift. This stock price model was used by Kim and Omberg (1969) who managed to obtain closed form solutions for the optimal investment strategies. Later, Watcher (2002) added intertemporal consumption to the model considered by Kim and Omberg. Furthermore, optimal strategies are obtained in closed form and an empirical analysis has shown that under some assumptions the stock drift is mean reverting for realistic parameter values. We solve within this framework the Hamilton–Jacobi–Bellman (HJB) equation associated with maximizing the expected utility

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of consumption, terminal wealth and legacy and this in turn provides the optimal investment, consumption and life insurance acquisition. By looking at the explicit solutions we found out that: (1) when the wage earner accumulates enough wealth (to leave to his/her heirs), he/she considers buying pension annuities rather than life insurance (2) when the value of human capital is small the wage earner buys pension annuities to supplement his/her income (3) when the wage earner becomes older (so the hazard rate gets higher) the wage earner has a higher demand (in absolute terms) for life insurance/pension annuities.

In our special model, due to the same risk preference for investment, consumption and legacy, the optimal legacy (wealth plus insurance benefit) is related to the optimal consumption. More precisely the ratio between the optimal legacy and optimal consumption is a deterministic function which depends on the hazard rate, risk aversion, premium insurance ratio, the weight on the bequest, and is independent of the financial market characteristics. In a frictionless market (i.e. hazard rate equals the premium insurance ratio) it turns out that the optimal legacy equals the optimal consumption.

Our main motivation in writing this paper is to explore the impact stochastic market price of risk has on the optimal investment strategies. This could not be observed by previous works which assumed a geometric Brownian motion model for the stock prices. We found out that the optimal investment strategy (in the stock) is significantly affected by the market price of risk (MPR). Moreover, the optimal investment in the stock is increasing in the MPR. As for the optimal insurance we found two patterns depending on the risk aversion. A risk averse wage earner pays less for life insurance if MPR is increasing up to a certain threshold; when the MPR exceeds that threshold, the amount the wage earner pays for life insurance starts to decrease. A risk seeking wage earner has a different behaviour; thus he/she pays less for life insurance when the MPR increases.

Organization of the paper: The remainder of this paper is organized as follows. In Section 2 we describe the model and formulate the objective. Section 3 performs the analysis. Numerical results are discussed in Section 4. Section 5 concludes. The paper ends with an appendix containing the proofs.

2. The model

In this paper, we assume that a wage earner has to make decisions regarding consumption, investment and life insurance/pension annuity purchase. Let $T > 0$ be a finite benchmark time horizon, and $\{W(t)\}_{t\in[0,T]}$ a 1-dimension Brownian motion on a probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathcal{F}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}$ is the completed filtration generated by $\{W(t)\}_{t\in[0,T]}$. Let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$. The continuous time economy consists of a financial market and an insurance market.

2.1. The financial market

The financial market contains a risk-free asset earning interest rate $r \geq 0$ and one risky asset. By some notational changes we can address the case of multiple risky assets. The asset prices evolve according to the following equations:

$$dB(t) = rB(t)dt, \qquad B(0) = 1,$$
$$dS(t) = S(t)\left[\mu(t)\,dt + \sigma\,dW(t)\right].$$

Moreover, the market price of risk $\{\theta(t)\}_{t\in[0,T]} = \{\frac{\mu(t)-r}{\sigma}\}_{t\in[0,T]}$ is a mean reverting process, i.e.,

$$d\theta(t) = -k(\theta(t) - \bar{\theta})dt - \sigma_\theta dW(t).$$

Here $\sigma, \sigma_\theta, k, \bar{\theta}$ are positive constants. This model for the stock price was considered by Kim and Omberg (1969) and Watcher (2002). In addition, we assume that the wage earner receives income at the random rate i which follows a geometric Brownian motion

$$di(t) = i(t)(\nu_i dt + \sigma_i dW(t)),$$

with positive constants ν_i and σ_i.

2.2. The life insurance

We assume that the wage earner is alive at $t = 0$ and his/her lifetime is a non-negative random variable τ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of the Brownian motion $\{W(t)\}_{t\in[0,T]}$. Let us introduce the hazard function $\lambda(t) : [0, T] \to \mathbb{R}^+$, that is, the instantaneous death rate, defined by

$$\lambda(t) = \lim_{\varepsilon\to 0} \frac{\mathbb{P}(t \leq \tau < t+\varepsilon \mid \tau \geq t)}{\varepsilon}.$$

From the above definition it follows that:

$$\mathbb{P}(\tau < s \mid \tau > t) = 1 - \exp\left\{-\int_t^s \lambda(u)\,du\right\}, \tag{2.1}$$

and

$$\mathbb{P}(\tau > T|\tau > t) = \exp\left\{-\int_t^T \lambda(u)\,du\right\}. \tag{2.2}$$

Denote by $f(s; t)$ the conditional probability density for the death at time s conditional upon the wage earner being alive at time $t \leq s$. Thus

$$f(s; t) = \lambda(s)\exp\left(-\int_t^s \lambda(v)dv\right). \tag{2.3}$$

Here $F(s; t)$ denotes the conditional probability for the wage earner to be alive at time s conditional upon being alive at time $t \leq s$; consequently

$$F(s; t) = \exp\left(-\int_t^s \lambda(v)dv\right). \tag{2.4}$$

In our model the wage earner purchases term life insurance/pension annuity with the term being infinitesimally small. Premium is paid (life insurance) or received (pension annuity) continuously at rate $p(t)$ given time t. In compensation, if the wage earner dies at time t when the premium payment rate is $p(t)$, then either the insurance company pays an insurance amount $\frac{p(t)}{\eta(t)}$ if $p(t)$ is positive, or the amount $\frac{p(t)}{\eta(t)}$ should be paid by wage earner's family if $p(t)$ is negative. Here $\eta : [0, T] \to \mathbb{R}^+$ is a continuous, deterministic, prespecified function called premium-insurance ratio, and $\frac{1}{\eta(t)}$ is referred to as loading factor. In a frictionless market $\eta(t) = \lambda(t)$, but due to commission fees $\eta(t) > \lambda(t)$. In order to simplify the analysis we assume that $\eta(t) = \lambda(t)$.

The insurance market model was pioneered by Yaari (1965) who considered the problem of optimal financial planning decisions for an individual with an uncertain lifetime. Later, that model was extended by Richard (1975). Many works followed Richard (1975) and Yaari (1965) by considering instantaneous term life insurance; it means that the investor can only purchase life insurance for the next instant; if surviving the next instant the investor has to buy again the instantaneous term life insurance and so on. The purchase of an annuity could be reversed by buying a life insurance that guarantees the payment of annuity's face amount on the death of the holder. Indeed there are situations of life annuities when premium is paid only at death (e.g. reverse mortgage). This brings up the notion of instantaneous pension annuity as the reverse of instantaneous term life insurance.