Insurance: Mathematics and Economics 51 (2012) 624–631

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Stochastic evaluation of life insurance contracts: Model point on asset trajectories and measurement of the error related to aggregation

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ARTICLE INFO

Article history:
Received July 2010
Received in revised form July 2012
Accepted 3 September 2012

Keywords:
Life Insurance contracts
Unit-linked contracts
Embedded options
TMG guarantee
ALM
Stochastic models
Monte-Carlo simulation

ABSTRACT

In this paper,[1] we are interested in the optimization of computing time when using Monte-Carlo simulations for the pricing of the embedded options in life insurance contracts. We propose a very simple method which consists in grouping the trajectories of the initial process of the asset according to a quantile. The measurement of the distance between the initial process and the discretized process is realized by the L2-norm. L2 distance decreases according to the number of trajectories of the discretized process. The discretized process is then used in the valuation of the life insurance contracts. We note that a wise choice of the discretized process enables us to correctly estimate the price of a European option. Finally, the error due to the valuation of a contract in Euro using the discretized process can be reduced to less than 5%.

1. Introduction

The implementation of an asset/liability management (ALM) model for the management and economic capital evaluation of life insurance contracts requires a very important volume of computations within the framework of Monte Carlo simulations. Indeed, for each trajectory of the asset, the whole of the liability must be simulated, because of the strong interactions between the asset and the liability through the ratchet and through the redistribution of the financial and technical results (*cf.* Planchet et al. (2011)). This leads to the well known problem of nested simulations (*cf.* Bauer et al. (2010) and Gordy and Juneja (2008)).

Various approaches were developed to overcome the practical difficulty of implementing the nested simulation approaches, among which the most used are optimizations inspired from the importance sampling (*cf.* Devineau and Loisel (2009)) and the techniques of replicating portfolio (*cf.* Revelen (2009), Schrager (2008) and Chauvigny and Devineau (2011)). More recently, Bauer et al. (2010) have used the LSMC approach initially proposed by Longstaff and Schwartz (2001) for the pricing of American options. However, optimization techniques are conceived generally for the estimation of the quantile of the excess asset/liability in the framework of the determination of the economic capital and are not always suited to compute the best estimate of the provision. Replicating portfolio approaches are wrongly adapted to the context of French insurance life contracts because of the complexity required when implementing clauses of redistribution of the financial discretionary benefit.

Therefore, practitioners sometimes use a method consist in summarizing the possible evolutions of the asset process in a limited number of characteristic trajectories. This results in proposing a limited number of scenarios of evolution for the asset process, each of these scenarios being characterized by a probability of occurrence. The difficulty is to build the scenarios in an optimal way in order to obtain a good approximation of the value of the provision.

The objective of this paper is to propose a method to build these characteristic trajectories and to provide tools to measure the impact of this simplification on the results. So we provide here a tool for best estimate computing which can be used together with other optimization techniques.

To achieve this goal in an objective manner, we propose a simple discretization of the distribution of the underlying trajectories in an L^2 Hilbert space. Many papers deal with the question of the time discretization of the path of the process (see for example the work of Gobet (2003) and the numerous references

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[1] Version of 2012/07/08.

0167-6687/$ – see front matter
doi:10.1016/j.insmatheco.2012.09.001

therein) and the question of the bias reduction. We will adopt in this paper a different point of view and focus on the discretization of the distribution of the paths. More precisely, a stochastic process S such as those considered here can be viewed as a random variable in an L^2 space. The probability distribution of S is in practice considered as continuous. What we want to do is to find a discrete probability distribution that is "not too far" from the true one. We do not think there is many works on this topics.

2. General characteristics of the discretized process

2.1. Definition

We consider a stochastic process $S(t)$ in $\Omega = [0, +\infty[,]-\infty, +\infty[, \ldots$ observed on the time interval $[0, T]$. In practice, $S(t)$ can represent a market value or a total return of assets portfolio. We replace the sample of trajectories of this process by the following simplified trajectories:

- at time t, we choose a partition of Ω, $\{[s_{t,j-1}, s_{t,j}[, 1 \le j \le p\}$;
- we then write

$$\xi_j(t) = \mathbf{E}(S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[); \tag{0.1}$$

- we define the process $\xi(t)$ by selecting one of the p trajectories of $\xi_j(t)$, each trajectory being characterized by its probability $\pi_{t,j} = \Pr(S(t) \in [s_{t,j-1}, s_{t,j}[)$.

Technically speaking, we replace the continuous distribution of the random variable S, which takes its values in a set of functions, by the discrete distribution of the variable ξ. We then make a package of trajectories according to the quantiles of the initial process $S(t)$. For example, we can choose the intervals so that $\pi_{t,j} = \frac{1}{p}$, which is the approach retained in this paper. In practice, we generally simulate trajectories of initial process S, $S_i(t)$, $1 \le i \le N$, and we usually estimate $\mathbf{E}(S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[)$ by the following estimator:

$$\tilde{\xi}_j(t) = \frac{1}{N_j} \sum_{i \in \Omega_j} S_i(t)$$

where $\Omega_j = \{i / S_i(t) \in [s_{t,j-1}, s_{t,j}[\}$ and $N_j = |\Omega_j|$.

Two errors are being made with this approximation:

- first, when replacing the trajectories of the continuous process $S(t)$ by the discretized process $\xi(t)$ obtained by selecting one of the p trajectories $\xi_j(t)$ with its probability $\pi_{t,j} = \Pr(S(t) \in [s_{t,j-1}, s_{t,j}[)$;
- second, the method of construction per simulation leads to replacing the theoretical expectation by an empirical estimation which introduces sampling fluctuations.

Generally speaking, this approximation is made within the framework of the valuation of options in life insurance contracts, and the projections are thus made under the risk neutral probability, which we shall suppose henceforward. In this context, for $r \ge 0$ interest free rate, $t \to e^{-rt}S(t)$ is a martingale under the risk neutral probability. In this paper, we are interested in the properties of the discretized process $\xi(t)$, which we call the discretized process associated with $S(t)$. We try to quantify and minimize the error generated by using this process to pricing embedded options of life insurance contracts.

First of all, we are going to study some characteristics of the discretized process $(\xi(t))_{t \le T}$. We begin by estimating distribution of this process. This distribution is established in a general context not requiring knowing characteristics of the initial process $S(t)$ (Section 2.2). The L^2-norm between the initial process and the discretized process gives a first vision of the error due to usage of the discretized process $\xi(t)$ (Section 2.3).

2.2. Distribution of the discretized process

$\xi(t)$ is a discrete process because it can take a finite number of values. Indeed, $\xi(t)$ takes p possible values $\{\xi_j(t), i = 1 \ldots p\}$ with probabilities $\{\pi_{t,j}, i = 1 \ldots p\}$. We note that $\xi_j(t)$ are deterministic because the partitions of Ω, $\{[s_{t,j-1}, s_{t,j}[, 1 \le j \le p\}$ are not random. The process $\xi(t)$ is well defined and the mean of the random variables $\xi(t)$ is identical to the mean of the initial process $S(t)$ at the same time: $\mathbf{E}(\xi(t)) = \mathbf{E}(S(t))$.

Proof. When $\mathbf{E}(X|A) = \frac{1}{\Pr(A)}\mathbf{E}(X 1_A)$, we can write:

$$\begin{aligned}
\mathbf{E}(\xi(t)) &= \sum_{j=1}^{p} \pi_j \xi_j(t) \\
&= \sum_{j=1}^{p} \pi_j \mathbf{E}(S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[) \\
&= \sum_{j=1}^{p} \pi_j \left(\frac{1}{\Pr(S(t) \in [s_{t,j-1}, s_{t,j}[)} \mathbf{E}\left(S(t)\mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[}\right) \right).
\end{aligned}$$

By using the definition $\pi_{t,j} = \Pr(S(t) \in [s_{t,j-1}, s_{t,j}[)$ we have

$$\begin{aligned}
\mathbf{E}(\xi(t)) &= \mathbf{E}\left(S(t) \sum_{j=1}^{p} \mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[} \right) \\
&= \mathbf{E}\left(S(t) \mathbf{1}_{\bigcup_{j=1}^{p} \{S(t)\in[s_{t,j-1},s_{t,j}[\}} \right) = \mathbf{E}(S(t))
\end{aligned}$$

because $\{[s_{t,j-1}, s_{t,j}[, 1 \le j \le p\}$ is a partition of Ω. □

2.3. L^2-norm between $\xi(t)$ and $S(t)$

We want to estimate the L^2-norm between the initial process and the discretized process. This norm is defined by:

$$\|S - \xi\|_{L^2} = \mathbf{E}\left(\int_0^T (S(t) - \xi(t))^2 dt \right)^{\frac{1}{2}}. \tag{0.2}$$

It can be correctly calculated by using

$$\begin{aligned}
\|S - \xi\|_{L^2} &= \sqrt{\sum_{j=1}^{p} \left(\int_0^T \mathrm{Var}(X_j(t)) dt \right)} \\
&= \sqrt{\int_0^T \sum_{j=1}^{p} \mathrm{Var}(X_j(t)) dt}
\end{aligned} \tag{0.3}$$

where $X_j(t) = S(t)|S(t) \in [s_{t,j-1}, s_{t,j}[$. The proof of this result is shown below.

Proof. We are interested in calculating $\mathbf{E}(\int_0^T (S(t) - \xi(t))^2 dt)$. Because $\{[s_{t,j-1}, s_{t,j}[, 1 \le j \le p\}$ is a partition of Ω, the intersection of two distinct sets $\{S(t) \in [s_{t,j-1}, s_{t,j}[\}$ is empty. Thus, we can write

$$\begin{aligned}
&\mathbf{E}\left(\int_0^T (S(t) - \xi(t))^2 dt \right) \\
&\quad = \mathbf{E}\left(\int_0^T \left(\sum_{j=1}^{p} (S(t) - \xi_j(t)) \mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[} \right)^2 dt \right) \\
&\quad = \mathbf{E}\left(\sum_{j=1}^{p} \int_0^T (S(t) - \xi_j(t))^2 \mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[} dt \right) \\
&\quad = \sum_{j=1}^{p} \mathbf{E}\left(\int_0^T (S(t) - \xi_j(t))^2 \mathbf{1}_{S(t)\in[s_{t,j-1},s_{t,j}[} dt \right).
\end{aligned}$$