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Stochastic analysis of life insurance surplus

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HIGHLIGHTS

- The paper is concerned with the evaluation of life insurance surplus.
- A methodology to study stochastic behavior of surplus is proposed.
- The methodology is illustrated on homogeneous portfolios of life insurance policies.
- The analysis is done in the environment of stochastic mortality experience and interest rates.

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ABSTRACT

The aim of the paper is to examine the behavior of insurance surplus over time for a portfolio of homogeneous life policies. We distinguish between stochastic and accounting surpluses and derive their first two moments. A recursive formula is proposed for calculating the distribution function of the accounting surplus. We then examine the probability that the accounting surplus becomes negative in a given insurance year. Numerical examples illustrate the results for portfolios of temporary and endowment life policies assuming a conditional AR(1) process for the rates of return.

1. Introduction

The surplus is an important indicator of an insurance company's financial position. In this paper, we present a methodology for studying the insurance surplus for a homogeneous portfolio of life insurance policies in an environment of stochastic rates of return and mortality following a nonparametric life table. We restrict attention to a simplified framework, in which only the cash flows arising from the benefit and premium payments, driven by the mortality experience of the portfolio, are accounted for. The mortality experience of the portfolio is based on a nonparametric life table. Mortality improvements over time would represent an additional source of risk that could be modeled by a wide range of stochastic processes. See, for example, Lee and Carter (1992), Cairns et al. (2006) and Currie et al. (2006). Such models for mortality improvements are beyond the scope of this paper. Expenses and other possible sources of decrements (e.g., lapses) are ignored. In order to determine the value of cash flows at a given time point, we model the investment component via a global rate of return, which is assumed to follow a conditional autoregressive process. A similar approach has also been adopted in Parker (1997). Although our framework has some limitations, the results and conclusions are believed to be useful in enhancing actuaries' understanding of the stochastic behavior underlying life insurance products.

The study of life contingencies in a similar environment of stochastic mortality experience based on a life table and stochastic interest rates can be traced back to the 1970s, and by now there exists a vast actuarial literature on the topic. In particular, among the papers that consider portfolios of life insurance and life annuity contracts are Frees (1990), Parker (1994, 1996, 1997) and Coppola et al. (2003). Marceau and Gaillardetz (1999) look at the reserve calculation for general portfolios of life insurance

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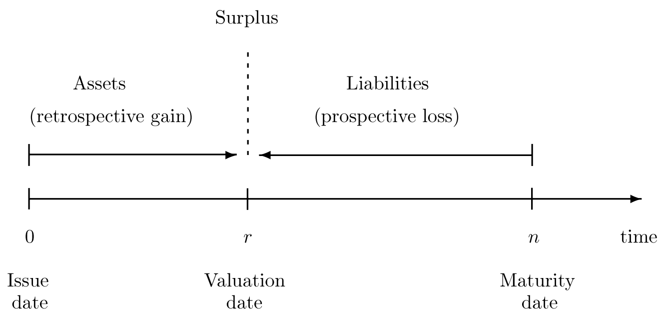

Fig. 1. A diagram to illustrate the definition of the stochastic surplus as the difference between the retrospective gain and the prospective loss at the valuation date (at time r) viewed from the issue date (at time 0).

policies and examine the appropriateness of using the limiting portfolio approximation. For an extensive literature review on the subject, the reader is referred to Hoedemakers et al. (2005); the latter paper proposes an approximation for the distribution of the prospective loss for a homogeneous portfolio of life annuities based on the concept of comonotonicity. In general, the above-mentioned papers deal with the stochastically discounted value of future contingent cash flows that are viewed and valued at the same point in time. This includes net single premium and reserve calculations.

The problem we try to address is of a different nature. To illustrate, consider a closed block of life insurance business at its initiation, time 0. At each future valuation date, we are interested in the financial position of this block of business as measured by the amount of the surplus available at that time. Let us fix one of the future valuation dates, say corresponding to time r, and consider how the surplus can be described at this valuation date. See Fig. 1 for a schematic illustration. Up to time r, the insurer will be collecting premiums and pay death benefits according to the terms of the contracts in the portfolio. The accumulated value to time r of these cash flows will represent the insurer's retrospective gain or the assets accumulated from this block of business. After time r, the insurer will continue to pay benefits as they come due and receive periodic premiums. This stream of payments, viewed at time 0 and discounted to time r, constitutes the prospective loss, a random variable which represents net future obligations or liabilities of the insurer. This leads to what we define as the stochastic surplus, i.e. the difference between the retrospective gain and the prospective loss. We also consider an alternative definition of the surplus (referred to as the accounting surplus) with the liabilities given by the actuarial reserve as opposed to the random prospective loss itself. There is an analogy between our approach and the dynamic solvency testing. The latter involves a projection of a company's solvency position into the future under varying assumptions; see e.g. Charles (1994). In our approach, instead of considering several scenarios, we average over all possible scenarios by placing a distribution on them.

The paper is organized as follows. In Section 2 we set up our framework for studying homogeneous portfolios assuming random mortality experience and stochastic rates of return. In Section 3.1 we introduce two types of insurance surplus, and derive their first two moments. For insurance regulators it is important that insurance companies maintain an adequate surplus. The actuarial liabilities of an insurer are reported in its financial statements as an actuarial reserve calculated in accordance with the regulations. So, when monitoring insurance companies, the regulators actually look at what we call here the accounting surplus. In Section 3.2, we propose a formula for obtaining the distribution function (df) of the accounting surplus that will appear in the financial statement of the company at a given future valuation date. One piece of information that is readily available from this df is the probability that the surplus falls below zero. If this probability is high, say above 5%, then the insurer may want to look into ways of improving the financial position associated with the given portfolio. Based on our set-up, possible mitigation measures could include an increase in the premium rate or raising additional initial surplus. Finally, in Section 4, numerical examples for two life insurance portfolios, one of endowment and the other of temporary policies, are used to illustrate main results of the paper. Conclusions and questions for future research are given in Section 5.

2. Framework

In the present paper, we consider *homogeneous portfolios*, that is, portfolios of identical life policies issued to a group of m policyholders all aged x with the same risk characteristics. It is assumed that the future lifetimes of the policyholders in the portfolio are independent and identically distributed. Each policy pays a death benefit b at the end of the year of death if death occurs within n years since the policy issue date and a pure endowment benefit c if the policyholder survives to the end of year n. The annual level premium π is payable at the beginning of each year as long as the contract remains in force.

2.1. Cash flows

To study the surplus of a portfolio, one approach would be to model the surplus for a single policy and then sum over the individual policies in the portfolio. However, this approach can be computationally time-consuming when dealing with large portfolios. To avoid such difficulties, instead we model aggregate annual cash flows and then sum over policy-years.

Consider a valuation date corresponding to time $r < n$; c.f. Fig. 1. For the purpose of our discussion of surplus and solvency, we distinguish between those cash flows that occur prior to time r and those that occur after. The cash flows prior to time r are defined as the net inflows into the company, whereas the cash flows after time r as the net outflows. This is consistent with the reporting of assets and liabilities.

Define the following indicator variables:

$$\mathscr{L}_{i,j} = \begin{cases} 1 & \text{if policyholder } i\ (i = 1, \dots, m) \text{ is alive at time} \\ & j\ (j = 0, \dots, n), \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathscr{D}_{i,j} = \begin{cases} 1 & \text{if policyholder } i\ (i = 1, \dots, m) \text{ dies in policy-year} \\ & j\ (j = 1, \dots, n), \\ 0 & \text{otherwise.} \end{cases}$$

For a homogeneous portfolio of m policies, let $\mathscr{L}_j := \sum_{i=1}^m \mathscr{L}_{i,j}$ and $\mathscr{D}_j := \sum_{i=1}^m \mathscr{D}_{i,j}$. That is, $\mathscr{L}_j$ is the number of people from the initial group of m policyholders who survive to time j (i.e., the number of in-force policies at time j), and $\mathscr{D}_j$ is the number of deaths in year j. Then, for the valuation at time r, the net cash inflow or the (annual) *retrospective cash inflow* at time j $(0 \le j \le r)$ is given by

$$\begin{aligned} RC_j^r &= \sum_{i=1}^m \left[\pi \cdot \mathscr{L}_{i,j} \cdot \mathbf{1}_{\{j<r\}} - b \cdot \mathscr{D}_{i,j} \cdot \mathbf{1}_{\{j>0\}} \right] \\ &= \pi \cdot \left(\sum_{i=1}^m \mathscr{L}_{i,j} \right) \cdot \mathbf{1}_{\{j<r\}} - b \cdot \left(\sum_{i=1}^m \mathscr{D}_{i,j} \right) \cdot \mathbf{1}_{\{j>0\}} \\ &= \pi \cdot \mathscr{L}_j \cdot \mathbf{1}_{\{j<r\}} - b \cdot \mathscr{D}_j \cdot \mathbf{1}_{\{j>0\}}, \end{aligned} \tag{2.1}$$

where $\mathbf{1}_{\{\mathcal{A}\}}$ is an indicator function; it is equal to 1 if condition $\mathcal{A}$ is true and 0 otherwise. That is, RC_j^r is the difference between the incoming premiums π from everyone alive at time j $(\mathscr{L}_j)$ and the outgoing death benefits b to those who died in year j $(\mathscr{D}_j)$.