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Social security as Markov equilibrium in OLG models: A note

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ABSTRACT

I refine and extend the Markov perfect equilibrium of the social security policy game in Forni (2005) for the special case of logarithmic utility. Under the restriction that the policy function be continuous, instead of differentiable, the equilibrium is globally well defined and its dynamics always stable.

Forni (2005) analyzes an economy with two-period lived overlapping generations, capital accumulation, exogenous labor supply, and a proportional wage tax whose receipts fund a lump-sum transfer to retirees, i.e. a PAYGO system. He argues that in a median voter environment without commitment and trigger strategies, there exist equilibria in which political decision makers support strictly positive social security taxes because they expect future social security benefits to be a decreasing function of the capital stock.[2] Under the restriction that the policy function be differentiable, the author characterizes equilibria of this type supporting a strictly positive tax rate τ_t as long as the capital stock, k_t, moves inside an interval $[\underline{k}, \bar{k}]$ that depends on parameter values and satisfies $\underline{k} > 0$ and $\bar{k} < \infty$.

The objective of this note is to refine and extend the results in Forni (2005) that correspond to the special case of logarithmic utility and Cobb–Douglas production function.[3] First, I derive conditions for a stable steady state to exist, requiring stronger restrictions on parameters. Second, for some $k_0 \in [\underline{k}, \bar{k}]$ the social security system collapses in finite time, and therefore the proposed policy function cannot be a solution for the entire interval $[\underline{k}, \bar{k}]$. Finally, requiring that the policy function be continuous, but not necessarily differentiable, I extend the domain of the policy function to $k_t \in [0, \infty)$.

The two key dynamic equations in Forni (2005) are:

$$\tau^F(k) = \frac{\alpha}{1-\alpha}\left(Ck^{-\frac{1+\alpha\beta}{1+\beta}} - 1\right), \tag{1}$$

$$g(k_{t+1}) \equiv \beta k_{t+1} + Ck_{t+1}^{1-\theta} = \frac{\beta}{1+n}\left(k_t^{\alpha} - \alpha C k_t^{\alpha-\theta}\right) \equiv h(k_t). \tag{2}$$

The first equation gives the equilibrium policy function under the restriction that this be a differentiable function of k, with β being the time discount factor, α the share of capital in production, and $C \geqslant 0$ a constant of integration. The second

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[2] Further work on Markov perfect equilibrium in economies with social security include Chen and Song (2009), Gonzalez-Eiras and Niepelt (2008), and Mateos-Planas (2008).

[3] Forni (2005) also solves the model numerically for the case of general CRRA preferences.

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equation, in which n is the net rate of population growth and $\theta \equiv (1+\beta\alpha)/(1+\beta)$, results after substituting the policy function τ^F into the capital accumulation expression

$$(1+n)k_{t+1} = \frac{1-\alpha}{1+\beta}\left[\beta(1-\tau_t)k_t^{\alpha} - \frac{\tau_{t+1}k_{t+1}(1+n)}{\alpha}\right]. \tag{3}$$

The restriction that $\tau_t \in (0,1)$ leads to the constraint that (1) and (2) only hold for $k_t \in (\underline{k} = (\alpha C)^{1/\theta}, \bar{k} = C^{1/\theta})$, where $\tau^F(\underline{k}) = 1$ and $\tau^F(\bar{k}) = 0$. Two properties of (2) that I will repeatedly use are that the relation between k_t and k_{t+1} is continuous even if it has no closed form solution, and that $g'(\cdot)$ and $h'(\cdot)$ are continuous functions. Furthermore, the relation between k_t and k_{t+1} is continuous at $\bar{k}$ since taxes become zero at this point. Finally, since $\lim_{k\to\underline{k}} g(k) > 0$, $\lim_{k\to\underline{k}} h(k) = 0$, $\lim_{k\to\infty} g'(k) = \beta > 0$, and $\lim_{k\to\infty} h'(k) = 0$, if there exists a stable steady state, k_s^{ss}, there also exists an unstable steady state k_u^{ss}, with $\underline{k} < k_u^{ss} < k_s^{ss}$.[4,5]

I start by deriving conditions for a stable steady state to exist. When $C = 0$ the dynamics of capital reduce to the case of an economy without taxes which has a unique stable steady state. Increases in C shift the function $g(k)$ upwards and $h(k)$ downwards. Thus, there exists a $\bar{C}$ such that for $C > \bar{C}$ the economy has no non-trivial steady state. There is no closed form expression for $\bar{C}$ but it can be characterized by[6]

$$\bar{C} = \inf\{C \text{ s.t. } g(k) \leqslant h(k)\ \forall k\}.$$

The restriction $C \leqslant \bar{C}$ is not sufficient to guarantee that at a stable steady state a positive social security tax rate is sustained. For this we also need that $k_s^{ss} < \bar{k}$. Otherwise the economy would accumulate, in finite time, a level of capital $k_t > \bar{k}$ for which there are no intergenerational transfers. By backward induction, no positive tax rates would be supported during the transition. If a steady state exists, a necessary and sufficient condition for $k_s^{ss} < \bar{k}$ is

$$g(\bar{k}) \geqslant h(\bar{k}). \tag{4}$$

To prove this claim, suppose to the contrary that $g(\bar{k}) < h(\bar{k})$. This implies that when $k_t = \bar{k}$ then $k_{t+1} > k_t$, and therefore $\tau_t = 0$. The dynamics in (3) with no taxes show that the capital stock would continue to increase until a steady state is reached, and thus $k_s^{ss} > \bar{k}$. Conversely, if a steady state exists and (4) holds, then since locally around $\bar{k}$ $k_{t+1} < k_t$, it cannot be that $k^{ss} > \bar{k}$. This follows since for $k > \bar{k}$ the dynamics corresponds to an economy without taxes, and thus $k_{t+1} < k_t$ holds $\forall k_t > \bar{k}$. Replacing $\bar{k} = C^{\frac{1}{\theta}}$ in (4) results in the following lower bound on C

$$C \geqslant \underline{C} \equiv \left(\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right)^{\frac{\theta}{1-\alpha}}.$$

Additionally, we need $\underline{C} \leqslant \bar{C}$, otherwise a stable steady state that supports a positive social security tax does not exist. These restrictions put bounds on the parameter C for which Forni's proposed policy function, (1), is indeed an equilibrium.

I now show that the lower end of the proposed domain, $\underline{k} = (\alpha C)^{1/\theta}$, is not correct. This value satisfies $h(\underline{k}) = 0$, and thus $k_{t+1} = 0$ from (2). Since $\underline{k} > 0$ (and therefore $0 = k_{t+1} < k_t = \underline{k}$) this means, by continuity of (2), that in an interval around $\underline{k}$ the capital stock is decreasing in time ($k_{t+1} < k_t$). From before, if a stable steady state exists then, by continuity of (2), and of $h'(\cdot)$ and $g'(\cdot)$, there must exist an unstable steady state and $\underline{k} < k_u^{ss} < k_s^{ss}$. Thus $k_{t+1} < k_t$ when $k_t \in [\underline{k}, k_u^{ss})$. Therefore if $k_0 \in [\underline{k}, k_u^{ss})$ then, in finite time, $k_t < \underline{k}$ and the social security system would collapse. By backward induction there cannot be a positive tax rate along these paths.[7]

We can therefore restate the results in Forni (2005) by saying that as long as $\underline{C} \leqslant \bar{C}$ there exist multiple expectational equilibria that sustain a PAYGO system, indexed by $C \in [\underline{C}, \bar{C}]$. His proposed solution for the policy function, (1), holds as long as the initial capital, k_0 satisfies $k_0 \in [k_u^{ss}, \bar{k}]$.[8]

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[4] This follows from the fact that a stable steady state is characterized by $h(k_s^{ss}) = g(k_s^{ss})$ and $h'(k_s^{ss}) < g'(k_s^{ss})$, and that $\lim_{k\to\underline{k}} h(k) < \lim_{k\to\underline{k}} g(k)$. By continuity of (2), and of $h'(\cdot)$ and $g'(\cdot)$, then there must exist a $k \in (\underline{k}, k_s^{ss})$ that satisfies $h(k) = g(k)$ and $h'(k) > g'(k)$, i.e. k is an unstable steady state.

[5] To rule out the existence of more than two steady states, a sufficient condition would be that $h''(k) < g''(k) < 0\ \forall k \geqslant \underline{k}$. If there are more steady states in general there are an even number of them. I will assume that at most two steady states exist. This assumption does not affect the results derived.

[6] Alternatively, under the assumption that at most two steady states exist, $\bar{C}$ can be characterized by the following system of two equations in the two unknowns, k, and $\bar{C}$,

$$g(k) = h(k),$$
$$g'(k) = \beta + \bar{C}(1-\theta)k^{-\theta} = \frac{\beta}{1+n}\left(\alpha k^{\alpha-1} + \alpha(\theta-\alpha)\bar{C}k^{\alpha-\theta-1}\right) = h'(k).$$

An upper bound on $\bar{C}$, $\bar{\bar{C}}$, can be found when $h''(k) < g''(k) < 0\ \ \forall k \geqslant \underline{k}$. This is done by solving for $g'(\underline{k}) = h'(\underline{k})$, and results in $\bar{\bar{C}} = \frac{1}{\alpha}\left[\frac{\beta^2(1+\alpha\beta)^2}{(1+n)\alpha(1+\beta)}\right]^{\frac{\theta}{1-\alpha}}$.

[7] Forni (2005) arrives to (1) and (2) under the restriction that *current* taxes are interior, $\tau_t \in (0,1)$ (see his Appendix A), while the correct formulation requires that this condition be satisfied by all *future* taxes as well (Forni acknowledges this when describing equilibrium dynamics).

[8] Since there are multiple equilibria, one should write $k_u^{ss}(C)$ and $\bar{k}(C)$. For simplicity the indexation on C is dropped.