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Dynamic programming approach to a Fermat type principle for heat flow

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Abstract

We consider nonlinear heat conduction satisfying a variational principle of Fermat type in the case of stationary heat flow. We review origins of a physical theory and transform it into a formalism consistent with irreversible thermodynamics, where the theory emerges as a consequence of the theorem of minimum entropy production. Applications of functional equations and the Hamilton–Bellman–Jacobi equation are effective when Bellman's method of dynamic programming is applied to propagation of thermal rays. Potential functions describing minimum resistance are obtained by analytical and numerical methods. For the latter, approximation schemes are developed. Differences between propagation of thermal and optical rays are discussed and it is shown that while simplest optical rays can be described by Riemmanian geometry, it is rather Finslerian geometry that is valid for thermal rays. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Consider a steady-state heat conduction in a rigid solid. When a thermal field is imposed by fixing the thermal gradient, the flow of thermal energy can be described in terms of 'thermal rays', the paths of heat flow determined by the direction of the temperature gradient and nonlinear properties of the conducting medium. When the thermal conductivity changes along the length of a thermal ray, the path along which the ray moves is, in general, curvilinear. Our purpose is prediction of the shapes of thermal rays, regardless of whether their curvilinearity is caused by the thermal inhomogeneity or material inhomogeneity of the medium. Here the thermal rays are shown to travel along paths satisfying the principle of minimum of entropy production which looks at first glance quite different from the well-known Fermat principle of minimum time (minimum optical length) for optical rays. However, taking into account that the minimum of entropy production is associated with the minimum resistivity of the path, it is easy to conclude that the minimum resistivity causes (in the dual problem) the maximum of heat flux through the medium or makes the residence time of heat in the medium as short as possible. This makes the principle for travel of thermal rays quite similar to that for propagation of light [6]. Our purpose is to investigate these phenomena by the method of dynamic programming [1] showing the similarities and differences between the optical and thermal phenomena.

2. Thermodynamics and propagation of steady thermal rays

Consider the entropy production functional describing the transfer of pure heat in a rigid solid under the assumption that the thermodynamic Hamiltonian vanishes, i.e. the rate dependent dissipation function Φ

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Nomenclature

A	variable area perpendicular to heat flow
A_0	constant area of transfer projected on axis y
c	bending constant for a thermal ray
H	Hamiltonian function
I	heat current through the area A
k	Onsager's conductivity related to gradient of T^{-1}
l	length parameter
n	refraction coefficient
p	momentum type integral, $\partial R/\partial y$
R	total resistance of thermal path
$R(x, y)$	minimum resistance potential
S_σ	entropy generated during a finite period of time
T	temperature
t	time
W^N	optimized performance function
ρ	thermal resistance as reciprocal of Onsager's conductance k
x	direction perpendicular to the resistivity gradient
y	direction tangent to the resistivity gradient

equals the state dependent dissipation function Ψ. This condition is associated with the requirement that an entropylike function generated along the kinetic paths is the true thermodynamic entropy which does not explicitly contain the time t

$$S_\sigma = \int_{t_1,\, v}^{t_2} L_\sigma \,\mathrm{d}V\,\mathrm{d}t \equiv \int_{t_1,\, v}^{t_2} (\Phi_s + \Psi_s)\,\mathrm{d}V\,\mathrm{d}t = \int_{t_1,\, v}^{t_2} 2\Phi_s \,\mathrm{d}V\,\mathrm{d}t = \int_{t_1,\, v}^{t_2} \rho J_q^{\,2}\,\mathrm{d}V\,\mathrm{d}t, \tag{1}$$

where $\Phi_s = \Psi_s$ [5]. The symbol ρ designates the reciprocal of the well-known Onsager's coefficient k for the heat conduction. This reciprocal has the meaning of the specific resistance for heat transfer, hence its designation. The energy is transferred along the length dl by the cross-section perpendicular to the heat flux. The perpendicular crosssection has the area A which may change with l; the volume differential $\mathrm{d}V = A\,\mathrm{d}l$. As distinguished from more standard treatments, we integrate here over the volume V 'moving with the energy'; in this case, x and y are special Lagrange coordinates and the heat flow is attributed to motion of the same portion of energy rather than to flow through a fixed area in the space. We introduce the heat current $I = \mathrm{d}Q/\mathrm{d}t$ as the amount of the thermal energy received by the system per unit time. The heat Q is positive when it is added to the system. Then the heat flux densities satisfy $J_q = \mathrm{d}Q/A\mathrm{d}t$ or $J_q = I/A$, hence

$$S_\sigma = \int_{t_1, v}^{t_2} \rho\left(\frac{\mathrm{d}Q}{A\mathrm{d}t}\right)^2 \mathrm{d}V\,\mathrm{d}t = \int_{t_1,\, l}^{t_2} (\rho \mathrm{d}l)\left(\frac{\mathrm{d}Q}{A\mathrm{d}t}\right)^2 A\,\mathrm{d}t = \int_{t_1,\, l}^{t_2} \left(\frac{\rho}{A}\,\mathrm{d}l\right)\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2 \mathrm{d}t. \tag{2}$$

As ρ has the meaning of the specific thermal resistance, the differential expression

$$\mathrm{d}R \equiv \rho \frac{\mathrm{d}l}{A} \tag{3}$$

defines the first differential of the total resistance R. The total resistance itself is the path integral

$$R \equiv \int_{l_1}^{l_2} \frac{\rho}{A}\,\mathrm{d}l \tag{4}$$

The quantity R increases with the total length l and decreases with the cross-sectional area A. With this definition, Eq. (2) can be transformed in to a popular form that describes the generation of the Joule heat within a conductor. Indeed, as shown by Eq. (5) below, in the frame of the variable Q the entropy production is

$$S_\sigma = \int_{l_1}^{l_2} \left(\frac{\rho}{A}\right) \mathrm{d}l \int_{t_1}^{t_2} \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2 \mathrm{d}t = \int_{Q_1}^{Q_2} RI\,\mathrm{d}Q = \int_{Q_1}^{Q_2} \left(T_2^{-1} - T_1^{-1}\right) \mathrm{d}Q = \int_{t_1}^{t_2} RI^{\,2}\,\mathrm{d}t \tag{5}$$

This shows that the difference of thermal potentials between the two subsystems 1 and 2, $\Delta T^{-1} = 1/T_2 - 1/T_1$, causes the flow of the thermal energy $\mathrm{d}Q \equiv \mathrm{d}Q_2$ along the total resistance R to heat the subsystem 2. The Ohm's law for heat conduction holds in the form

$$I \equiv -\frac{\mathrm{d}Q_1}{\mathrm{d}t} = \frac{\mathrm{d}Q_2}{\mathrm{d}t} = \frac{\Delta T^{-1}}{R} = \frac{T_2^{-1} - T_1^{-1}}{R} \tag{6}$$

At the steady state

$$S_\sigma = \int_{Q_1}^{Q_2} \left(T_2^{-1} - T_1^{-1}\right) \mathrm{d}Q = \int_{t_1}^{t_2} RI^{\,2}\,\mathrm{d}t = \left(T_2^{-1} - T_1^{-1}\right) Q \tag{7}$$

Note that for an unsteady state process, when two bodies exchange heat and the system is isolated as the