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Dynamic Programming Method in the Generalized Traveling Salesman Problem: The Influence of Inexact Calculations

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Abstract—For the generalized traveling salesman problem, the known dynamic programming method (DPM) under conditions of inexact calculations of the Bellman function is investigated. Concrete estimates of the errors under the realization of the global extremum are obtained. In the scope of the basic problem, the important question about the sequential circuit of the sections of set-valued mappings is considered.

Keywords—Dynamic programming, Bellman function, Route optimization.

1. INTRODUCTION

This paper is the natural continuation of [1]. Here DPM of the route optimization is investigated. It is possible to represent many applied problems for which the necessity of the order of a finite system of sets should be optimized in the conjunction with a natural "tracing" of permutations. Now consider a hypothetical problem about the sequential circuit of an archipelago. Let N be a natural number $N \geq 2$ and $M_1, \dots, M_N$ are given islands. Fix an initial state $x^0 \in \mathbb{R}^2$. We choose a permutation $J = (i_j)_{j \in \overline{1,N}}$ of $\overline{1,N}$ and a system points $(x_i)_{i \in \overline{1,N}}$ on $\mathbb{R}^2$ for which $x_1 \in M_{i_1}, \dots, x_N \in M_{i_N}$. We call J the route in the generalized salesman problem. The system of permutations

$$\left(x_0 = x^0\right) \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_N \tag{1.1}$$

is a trajectory with respect to route J. We use J and system (1.1) with the following goal. Namely, we strive to minimize the sum of lengths $\|x_i - x_{i-1}\|$, $i \in \overline{1,N}$. As a corollary, we obtain the concrete problem of a discrete-continuous optimization. It is possible to construct a solution of this problem by the scheme of [2,3] that is a development of the natural approach ascending to [4,5]. But under this development, many difficulties arise. These difficulties are connected with calculation in the discrete-continuous extremal problems arising in the Bellman equation. Of course, the process of the determination of the Bellman function is accompanied by inevitable errors. The above-mentioned difficulties reduce to new approaches using different approximate constructions. In [6], one construction operating with the analysis of displacements with respect to varying subsets of $M_1, \dots, M_N$ is considered (note that in [6], another profound problem was considered). As a result, a problem of the circuit of sections of set-valued mappings arises. Of

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course, it is possible to recommend other informative settings generating the above-mentioned mathematical problem (note the more general setting of [7], too). In particular, we note the problem about the route choice in the space of dynamic structures. The realization of each concrete structure determines the natural possibility for the passage to points of the attainability domain corresponding to the chosen dynamic system. But, we return to the above-mentioned problem about sequential circuit of the islands $M_1, \dots, M_N$. The most obvious approach to the solution is defined on the basis of principle of the passage to the nearest point of a next set (island). This principle determines the choice of the above-mentioned next set, too. It is possible to construct on this basis the corresponding version of DPM (see [8]). But even in simple examples, the situations arise, for which refusing from the above-mentioned principle of nearest passages improves a result. It is possible to realize metric projection on $M_1, \dots, M_N$ approximately. As a corollary, some subsets of the above-mentioned islands are realized. And what is more, these subsets are transposed depending on a point corresponding to the visit place of the previous island. The above-mentioned sets connected with the operation of the metric projection play the role of attainability domains in control theory [9–14]. Of course, it is possible to consider these domains as sections of some set-valued mappings. The last approach is realized in [1]. This investigation is connected with the development of the basic method of [1] in "real conditions".

2. ELEMENTS OF THE METHOD OF DYNAMIC PROGRAMMING

Briefly recall the settings of [1], fixing a nonempty set X and the number $N \in \mathcal{N} \triangleq \{1; 2; \dots\}$, $N \geq 2$. Denote by 2^X the family of all nonempty subsets of X. Fix N set-valued mappings

$$A_1 : X \longrightarrow 2^X, \dots, A_N : X \longrightarrow 2^X. \tag{2.1}$$

So, for the fixed number i and $x \in X$, we obtain the set $A_i(x)$. The given set is a subset of X. Of course, for each i we have the point-to-set mapping. This mapping characterizes our possibilities. We interpret $A_i(x)$ as an attainability domain corresponding to the fulfilment of the assignment with the number $i \in \overline{1, N}$ under the initial state $x \in X$. Under the given $x^0 \in X$, we choose a permutation $\mathcal{I} \triangleq (i_1, \dots, i_N)$ of $\overline{1, N}$ and the system of the displacements

$$(x_0 = x^0) \to (x_1 \in A_{i_1}(x_0)) \to (x_2 \in A_{i_2}(x_1)) \to \cdots \to (x_N \in A_{i_N}(x_{N-1})). \tag{2.2}$$

It is possible to consider (2.2) as a trajectory realized in the conformity with route $\mathcal{I}$. Set-valued mappings (2.1) define a dynamic of a hypothetical system generating the trajectory (2.2). Fix a finite sequence $\mathbf{c} \triangleq (c_1, \dots, c_N)$ of criteria

$$c_1 : X \times X \longrightarrow [0, \infty[, \dots, c_N : X \times X \longrightarrow [0, \infty[. \tag{2.3}$$

We use $\mathbf{c}$ (2.3) for the estimation of the trajectory (2.2) in the form of the sum of all numbers $c_{i_1}(x_0, x_1), c_{i_2}(x_1, x_2), \dots, c_{i_N}(x_{N-1}, x_N)$.

REMARK 2.1. In the case of the problem of Section 1, it is possible to set X in the form of the union of the sets $\{x^0\}, M_1, \dots, M_N$ (suppose that $x^0 \notin M_i$ under $i \in \overline{1, N}$). The mapping (2.1) is defined in the form of the constant sets $M_1, \dots, M_N$, respectively. For each of functions (2.3), here it is possible to suppose $c_i(x^{(1)}, x^{(2)}) = \|x^{(1)} - x^{(2)}\|$, where $\|\cdot\|$ is a norm of $\mathbb{R}^2$. For the optimization of the lengths sum, it is possible to use method of [2,3]. We have a very difficult problem. It is possible to use the approach planned in [6] for problems of radio engineering. Namely, if $x \in \mathbb{R}^2$ and M is a nonempty subset of $\mathbb{R}^2$, then denote by $\rho(x, M)$ the distance from x until M. Fixing numbers $\varepsilon_1 > 0, \dots, \varepsilon_N > 0$, we define here $A_i(x) \triangleq \{y \in M_i \mid \|x - y\| \leq \rho(x, M_i) + \varepsilon_i\}$, where $x \in X$ and $i \in \overline{1, N}$. In this model, we "overlook" at the state $x \in X$