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Reflections on Optimality and Dynamic Programming

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Abstract—Discrete models and continuous control systems are considered in regard to optimality of their trajectories. Some aspects of the principle of optimality [1, p. 83] are analyzed, and it is shown to imply *total optimality*, that is, the optimality of every part of an optimal trajectory. Certain autonomous systems with free admissible variations possess this property.

Nonautonomous optimal systems are not, in general, totally optimal, in which case the principle of optimality is not valid. A modification is proposed for the derivation of the main functional equation to demonstrate that dynamic programming and its functional equations are valid also in the case of nonoptimal remaining trajectories under a certain *contiguity condition* that is defined and analyzed in the paper.

Control systems with incomplete information or structural limitations on controls do not, in general, satisfy the contiguity condition. Control problems for such systems may have optimal solutions which, however, cannot be obtained by dynamic programming. This fact is shown in an example of a widely used engineering system for which an optimal trajectory has all its remaining parts nonoptimal and noncontiguous to the optimal trajectory. The paper presents theoretical justification of dynamic programming for contiguous systems that do not conform to the principle of optimality. Examples are presented to illustrate the results which open new avenues in modeling and optimization of general (not totally optimal) control systems. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In the ten-line section from [1, p. 83] we read

"3. The Principle of Optimality

In each process, the functional equation governing the process was obtained by an application of the following intuitive:

Principle of Optimality. An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The mathematical transliteration of this simple principle will yield all the functional equations we shall encounter throughout the remainder of the book. A proof by contradiction is immediate."

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This short statement has laid the foundation of a new direction in operations research, resulting in "a mathematical method called 'dynamic programming'.... In addition to having an impact on applied mathematics and physical sciences, dynamic programming is widely regarded as a major contributor to economic and business management." (From [2].)

In control literature, the mathematical transliteration of the above principle for deterministic processes is related mostly to the following types of functional equations that we reproduce from [1,3], using the simplest representations in authors' notations.

1.1. Discrete Process [3, Part IV, Ch. VIII, Section 8, formula (8.3)]

$$f_{n+1}(p) = \max_q [R(p,q) + f_n(T(p,q))], \qquad n = 0, 1, \dots; \quad f_0(\cdots) = 0, \tag{1.1}$$

where p is the initial state vector for $n = 0$, q is a decision (control) vector corresponding to the first n steps (transitions), $R(p,q)$ is the revenue obtained after the first transition $n = 1$, $T(p,q) = p^1$ is the state vector after one transition, and $f_n(p)$ is the maximum revenue obtained after n steps corresponding to an optimal policy q^o defined by the max-operator in (1.1). Optimal policy q^o need not be unique, however, the optimal value $f_n(p)$ is unique. Clearly, for some problems, e.g., expense minimization, one should write *min* in (1.1), but the structure remains the same. Usually, $q \in S$, $p \in D$, $T(p,q) \in D$, *for all* $q \in S$, *all* $p \in D$, where S and D are bounded closed regions [1, Ch. III, Section 4]; if this is not the case, then *max/min* should be replaced by *sup/inf* [3, Part IV, Ch. VIII, Section 8].

1.2. Continuous Process

In this case, the sequence $f_n(p)$ in (1.1) is replaced by a single cost function $f(p)$, and if $R(p,q)$ is included in $f_n(p)$, the simple functional equation is obtained [1, Ch. III, Section 4, equation (5)]

$$f(p) = \max_q f(T(p,q)). \tag{1.2}$$

1.3. New Formalization of the Calculus of Variations [1, Ch. IX]

For a problem of optimal control with variable final time

$$\max J(u) = \int_0^T F(x,u)\,dt, \qquad \frac{dx}{dt} = G(x,u), \qquad x(0) = c, \tag{1.3}$$

denote

$$\max J(u) = f(c,T), \tag{1.4}$$

and split the integral in (1.3) over two segments $[0,S]$ and $[S,T]$. Then, similarly to (1.1),(1.2) where $R(p,q)$ corresponds to the integral over $[0,S]$, we get according to the principle of optimality

$$f(c,T) = \max_{u[0,T]} \left[\int_0^S F(x,u)\,dt + \int_S^T F(x,u)\,dt \right] \tag{1.5a}$$

$$= \max_{u[0,S]} \left[\int_0^S F(x,u)\,dt + f(c(S), T-S) \right], \tag{1.5b}$$