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Optimal structural design under stochastic uncertainty by stochastic linear programming methods

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Abstract

In the optimal plastic design of mechanical structures one has to minimize a certain cost function under the equilibrium equation, the yield condition and some additional simple constraints, like box constraints. A basic problem is that the model parameters and the external loads are random variables with a certain probability distribution. In order to get reliable/robust optimal designs with respect to random parameter variations, by using stochastic optimization methods, the original random structural optimization problem must be replaced by an appropriate deterministic substitute problem. Starting from the equilibrium equation and the yield condition, the problem can be described in the framework of stochastic (linear) programming problems with 'complete fixed recourse'. The main properties of this class of substitute problems are discussed, especially the 'dual decomposition' data structure which enables the use of very efficient special purpose LP-solvers.

Keywords: Optimal structural design; Random structural parameters; Stochastic linear programming

1. Optimal plastic design under stochastic uncertainty

The limitations of elastic structural theory with respect to the assessment of actual structural resistance, hence of safety, is well known [1]. Indeed, to accept as valid the elastic behaviour up to the critical point of the mechanical structure means to identify the structural failure with first attainment of a critical stress condition in any point of the structure, i.e. to assume a weakest-link type of behaviour. This holds in practice in some cases, such as statically determinant structures or structures made of brittle material, but in most structures a stress redistribution allows the loads to grow further beyond the level corresponding to the first critical condition of elastic theory. Plastic collapse of the structure occurs [31,35] when the structure is converted into a mechanism by providing a suitable number and dispositions of plastic zones. Thus, according to limit analysis, collapse is identified with the development of plastic hinges, i.e. zones of material with stress in the yield limit condition, in such a number and location to allow a movement of the whole or a part of the structure without requiring deformation of the zones with stresses below the yield limit (transformation of the structure into a plastic collapse mechanism).

Limit analysis is concerned [13,14,31] with establishing the strength of a structure, i.e. its capacity for the supporting of loads. Hence, using the plastic ductility of structural materials in improving the design of structures, limit analysis is not concerned with deformation: it cannot therefore provide the load carrying capacity for a structure with elements that have a limited ductility or deformability, nor for a structure which becomes unstable because of the displacements induced by plastic deformation, see Ref. [31].

For elastic-perfectly plastic materials, the ultimate load condition corresponding to complete collapse of the structure can be obtained by means of plastic limit (collapse) analysis through application of either a pair of dual theorems, i.e. the static (safe or lower bound) theorem and the kinematic (unsafe or upper bound) theorem. Considering a structure made of ductile material, collapse does not occur if there exists a statically admissible stress field, i.e. a stress vector in equilibrium with the applied loads and not violating the strength inequality or yield condition at any point of the structure. Conversely, collapse occurs if a collapse mechanism exists such that the work done by the applied loads is larger than the internal plastic work.

Since the fundamental static and kinematic relations for limit (collapse) analysis appear [16], naturally as linear equations, and the yield condition is convex, plastic limit

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analysis and design problems may be formulated by a linear or a convex program. However, using linear approximates of the convex yield condition, plastic limit analysis and optimal design problems may be formulated always (approximatively) in the framework of linear programming (LP), see e.g. [3,15,36].

Having the powerful and most efficient LP tools, a remaining major difficulty in plastic structural analysis and optimal plastic design as well as in other branches of optimal structural design, e.g. optimal material layout, is the fact that the material coefficients, such as yield stresses in compression and tension, and the applied (external) loads, but also many other model parameters of the analysis or design problem, such as cost, weight coefficients, etc., are not given, fixed quantities in practice. Due to random variations of the material and the loadings, due to manufacturing and modelling errors and further random disturbances, these parameters must be modelled through random variables on a certain probability space, where the joint probability distribution of the random quantities can be assumed to be known. Thus, in plastic analysis and in optimal plastic design the given structural optimization, material optimization or topology optimization problem with random parameters must be replaced by an appropriate deterministic substitute problem using stochastic optimization methods, see e.g. [9,10,19,23]. Based on certain decision theoretical principles, substitute problems incorporate the random parameter variations in order to reach robust optimal designs, i.e. optimal designs which are insensitive with respect to random parameter variations. This goal can be achieved by minimizing the expected structural costs, e.g. the structural weight or volume, subject to constraints for the expected costs of structural failure; as a special case, here we have constraints for the probability of failure for the whole structure and/or certain parts of the structure.

1.1. Fundamental theorems from plastic limit analysis

From collapse load analysis, the following fundamental theorems for structures having rigid-plastic or elastic-perfectly plastic materials are well known [1,3,4,6,12,26]:

Static Theorem (ST) (Lower bound or safe theorem). *If any stress distribution throughout the structure can be found which is everywhere in equilibrium internally and balances the external loads, and at the same time does not violate the yield conditions, these external loads will be carried safely by the structure.*

By means of duality theory (Kuhn–Tucker theorem) the equivalence between the above (ST) and the following kinematic theorem can be established [3,6,27].

Kinematic Theorem (KT) (Upper bound or unsafe theorem). *Collapse occurs if a collapse mechanism, i.e. a pair of vectors $(u, \tilde{u})$ exists, fulfilling the compatibility condition, such that the work done by the external loads is larger than the corresponding internal plastic work.*

In limit analysis and in optimal plastic design of structures [5,7,25–28,32,35–37] we have therefore the following basic conditions (I), (II):

(I) *The equilibrium equation*

$$\mathbf{CF} = \mathbf{R}(= \lambda_0 \mathbf{R}_0). \tag{1}$$

Here, $\mathbf{C}$ is the $m \times n$ equilibrium matrix with rank $\mathbf{C} = m < n$, $\mathbf{R}$, $\mathbf{R}_0$ are external load vectors, and λ_0 is a load factor. Moreover, $\mathbf{F}$ denotes the vector of internal or member loads (axial forces, twisting and bending moments).

(II) *The yield condition*

For trusses [11] or if, for plane or spatial frames, no member load interactions are taken into account [11,15], then the yield condition can be represented by the following inequalities

$$\mathbf{F}^{\mathrm{L}}(\sigma^{\mathrm{L}}, \mathbf{X}) \le \mathbf{F} \le \mathbf{F}^{\mathrm{U}}(\boldsymbol{\sigma}^{\mathrm{U}}, \mathbf{X}), \tag{2}$$

where the bounds $\mathbf{F}^{\mathrm{L}}$, $\mathbf{F}^{\mathrm{U}}$ for the n-vector $\mathbf{F}$ of member loads depend on the r-vector $\mathbf{X}$ of design variables $\mathbf{X}_k$, $k = 1, \ldots, r$, and the vectors $\boldsymbol{\sigma}^{\mathrm{L}}$, $\boldsymbol{\sigma}^{\mathrm{U}}$ of yield stresses in compression (<0) and tension (>0).

If for plane or spatial frames member load interactions must be taken into account, then for each beam $i = 1, \ldots, B$ we have to consider the vectors of member loads [33] at the two ('negative', 'positive') ends of the beam

$$\mathbf{F}_i^- = \begin{pmatrix} t_i \\ m_i^- \end{pmatrix}, \qquad \mathbf{F}_i^+ = \begin{pmatrix} t_i \\ m_i^+ \end{pmatrix} \text{(for plane frames)} \tag{3a}$$

$$\mathbf{F}_i^- = \begin{pmatrix} t_i \\ \tau_i \\ m_i^- \\ \mu_i^- \end{pmatrix}, \qquad \mathbf{F}_i^+ = \begin{pmatrix} t_i \\ \tau_i \\ m_i^+ \\ \mu_i^+ \end{pmatrix} \text{(for spatial frames)} \tag{3b}$$

containing the axial force t_i and the bending moments m_i^-, m_i^+, respectively, for plane frames, and the axial force t_i, the twisting moment τ_i and two bending moments m_i^-, μ_i^-, m_i^+, μ_i^+, respectively, for spatial frames. Let

$$\mathbf{F}_{\mathrm{p}i} = \mathbf{F}_{\mathrm{p}i}(\boldsymbol{\sigma}_{\mathrm{y}}, \mathbf{X}) := \begin{pmatrix} N_{\mathrm{p}i}(\sigma_{\mathrm{y}}, X) \\ M_{\mathrm{p}i}(\sigma_{\mathrm{y}}, X) \end{pmatrix}, \tag{4a}$$

$$\mathbf{F}_{\mathrm{p}i} = \mathbf{F}_{\mathrm{p}i}(\boldsymbol{\sigma}_{\mathrm{y}}, \mathbf{X}) := \begin{pmatrix} N_{\mathrm{p}i}(\sigma_{\mathrm{y}}, X) \\ T_{\mathrm{p}i}(\sigma_{\mathrm{y}}, X) \\ M_{\mathrm{p}i}^{\mathrm{y}}(\sigma_{\mathrm{y}}, X) \\ M_{\mathrm{p}i}^{z}(\sigma_{\mathrm{y}}, X) \end{pmatrix}, \tag{4b}$$

respectively, denote the corresponding principal axial, twisting and bending plastic capacities of the ith beam, where $\mathbf{X}$ designates again the design r-vector and $\boldsymbol{\sigma}_{\mathrm{y}}$ is the vector of yield stresses $\boldsymbol{\sigma}_{\mathrm{y}i}$, $i = 1, \ldots, B$. Then, the