Available online at www.sciencedirect.com

SCIENCE DIRECT®

Computers and Structures 84 (2006) 873–887

Computers & Structures

www.elsevier.com/locate/compstruc

Limit analysis of masonry block structures with non-associative frictional joints using linear programming

M. Gilbert [a,*], C. Casapulla [b], H.M. Ahmed [a]

[a] Department of Civil and Structural Engineering, University of Sheffield, Mappin Street, Sheffield S1 3JD, UK

[b] Department of "Costruzioni e Metodi Matematici in Architettura", University of Naples Federico II, Via Monteoliveto, 3, 80134, Naples, Italy

Received 22 February 2005; accepted 1 February 2006
Available online 24 March 2006

Abstract

Although limit analysis has been found to be a valuable tool for analysing the stability of masonry gravity structures, modelling non-associative Coulomb sliding friction can be problematic. A simple iterative procedure which involves the successive solution of linear programming sub-problems is presented in the paper. Using the procedure a specially modified Mohr-Coulomb failure surface is adopted at each contact interface, with all failure surfaces updated at each iteration until a converged solution is obtained. The procedure is applied to problems from the literature and also to new, considerably larger, benchmark problems.

Keywords: Masonry structures; Limit analysis; Non-associative friction; Flow rule; Coulomb friction; Linear programming

1. Introduction

Kooharian and Heyman [1,2] were amongst the first to consider (vaulted) masonry block structures in the context of the plastic limit analysis theorems which emerged during the last century. Initially the assumptions were simple: constituent blocks possess infinite compressive strength, joints have zero tensile strength and sliding failures are not permitted. Whilst such a simple idealization is undeniably attractive in that it simplifies a hand or computer-based analysis, it is also problematic. For example, in the case of flat arches failure simply cannot occur without some sliding and/or crushing of the material.

Ignoring crushing at present, it is clearly necessary to study the mechanics of masonry structures assuming that sliding failures can occur. However, it is well-known that the bounding theorems of plastic limit analysis do not in general provide unique solutions for the collapse load factor if a non-associative flow rule is specified. Non-associative friction is of interest since real behaviour of a joint is generally non-associative, i.e. $\delta_n \neq \delta_t \tan\varphi$, where δ_n and δ_t are respectively the normal (dilatant) and tangential relative displacements between sliding surfaces at a masonry joint, and where φ is the angle of friction. Whilst in practice some dilatancy will be likely to occur when two rough blocks pass over each other, experimental evidence indicates that real joint behaviour is quite complex, with the amount of dilatancy being dependent on the micro-scale geometrical and mechanical features of the masonry joint [3]. Also, it is found that the angle of dilatant friction tends to reduce both with increasing relative tangential displacement and also under the action of increasing normal stresses, Fig. 1. Consequently, when a simpler yet conservative model of behaviour is required, it is common to assume a Coulomb friction sliding model. As indicated on Fig. 1, this model stipulates zero dilatancy. This model is assumed in the paper.

Drucker [4] was amongst the first to identify the difficulty of treating Coulomb sliding friction. He stated modified upper-bound conditions assuming either complete

* Corresponding author. Tel.: +44 (0)114 2225418; fax: +44 (0)114 2225700.
E-mail address: m.gilbert@sheffield.ac.uk (M. Gilbert).

0045-7949/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruc.2006.02.005

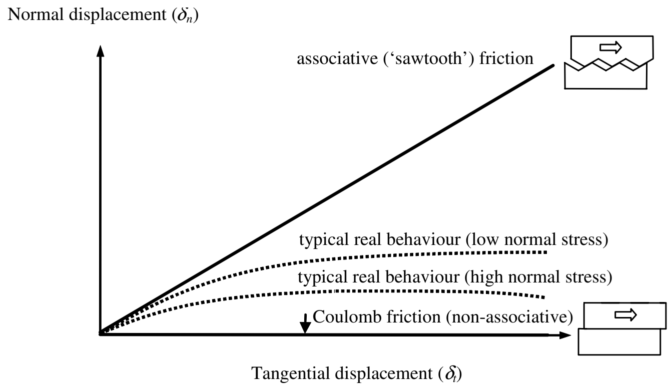

Fig. 1. Masonry joint behaviour: relationship between associative & Coulomb friction (non-associative) idealisations and typical real behaviour.

attachment or dilatancy and a lower-bound condition assuming a zero coefficient of friction. With reference to a generic non-standard material, other authors [5–8] provided lower and upper bounds on the exact solution by using Radenkovic's theorems. However, such bounds will often be too wide to be of use in practice. For example, if a zero coefficient of friction is used the collapse load factor for a simple masonry arch rib (a particular, but extremely common, assemblage of blocks) will typically be bounded from below by a load factor of zero.

More recently Livesley [9,10], using a static equilibrium approach, developed a formal linear programming (LP) procedure to compute the load factor for two and three-dimensional structures formed from rigid blocks, assuming initially an associative friction model. Since Charnes and Greenberg [11] had shown many years previously that when "mechanism" and "equilibrium" limit analysis formulations are linearised they give rise to dual LP problems, Livesley was also able to plot a collapse mechanism directly after performing a lower-bound analysis. In doing so, he identified apparently anomalous failure mechansisms, and also demonstrated cases when the associative friction load factor over-estimated the Coulomb sliding friction load factor. Consequentially he proposed a post-optimality analysis to test the validity of the solutions obtained (applicable to simple masonry vault problems), although no remedy was proposed in cases of load factor overestimation.

It should perhaps be mentioned that for certain classes of problems the assumption of associative friction has been found to provide numerical predictions which are in broad agreement with experimentally observed results. For example, in a study of the behaviour of multi-ring arches the first author [12] established the importance of failure modes involving sliding for this structural form and also observed that associative friction solutions appeared to agree reasonably well with experimental results.

Nevertheless, Livesley's pioneering initial study of non-associative friction stimulated a line of research [13–17] concerned with developing numerical procedures for such problems. One of these was given by Lo Bianco and Mazzarella [13], followed by Baggio et al. [14,15]. Here the non-associative problem was solved using procedures which involved identifying load factors simultaneously satisfying the kinematic and static conditions. However the procedures were found to be rather onerous in terms of time and memory requirements, because of the non-linear and non-convex optimisation procedures required.

Others have identified particular non-associative problems that are amenable to simplification and which will hence furnish safe solutions which are both statically admissible and which satisfy the normality rule [18–20]. The second author then identified important problem types, with particular loading and geometry conditions, for which unique solutions can be found [21,22]. For certain other problems, procedures have been suggested to get closer to the exact solution, moving from the bounds given by two associative friction kinematic models [23]. However, these lines of research are still far from providing a general method of analysis for non-associative problems of arbitrary geometry.

Most recently the problem has been posed as a mixed complementarity problem (MCP), and a mathematical programming with equilibrium constraints (MPEC) formulation has been proposed for masonry limit analysis problems involving non-associative frictional sliding [24,25]. Unfortunately relatively specialised non-linear programming solution methods must be employed and it also seems that solving the MPEC formulation in the way proposed may for practically large problems be prohibitively computationally expensive.

The inherently non-linear problem of analysing the stability of structures composed of rigid blocks in the presence of material crushing has been tackled with some success by using an approximate procedure which involves the solution of a series of successively modified LP problems [26]. A key aim of the present study is to determine whether a similar approach can be applied to problems involving non-associative friction (which are both non-linear and non-convex), building on initial studies made by the authors [27]. Breaking the problem down to the solution of a series of successively modified LP problems is attractive since modern interior-point based LP solvers are efficient, robust and now very widely available. Additionally such a solver is already used in the RING rigid block analysis software for masonry arch bridges, originated by the first author (www.shef.ac.uk/ring).

2. Discrete block model with frictional constraints

Since the specific bonding pattern of masonry structures often influences the failure mode and load factor, there is some justification for modelling masonry structures as assemblages of discrete blocks. This is assumed here, along with the following assumptions: constituent blocks are rigid and infinitely strong; no tension may be transmitted across joints; blocks may slide and/or rock relative to each other.