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Fuzzy Sets and Systems 157 (2006) 1283 – 1302
FUZZY sets and systems
www.elsevier.com/locate/fss

Duality in fuzzy linear programming with possibility and necessity relations

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Received 4 January 2005; received in revised form 27 November 2005; accepted 28 November 2005
Available online 4 January 2006

Abstract

A class of fuzzy linear programming (FLP) problems with fuzzy coefficients based on fuzzy relations is introduced, the concepts of feasible and (α, β)-maximal and minimal solutions are defined. The class of crisp (classical) LP problems can be embedded into the class of FLP ones. Moreover, for FLP problems a new concept of duality is introduced and the weak and strong duality theorems are derived. The introduced concepts and results are illustrated and discussed on a simple numerical example.

Keywords: Fuzzy relations; Fuzzy linear programming; Duality

1. Introduction

The problem of duality has been investigated since the early stage of fuzzy linear programming (FLP), see [3,10,12,1]. However, up till now the duality theory in FLP problems, i.e. LP problems with fuzzy coefficients have not been satisfactorily resolved. In this paper we first introduce a broad class of FLP problems and define the concepts of β-feasible and (α, β)-maximal and minimal solutions of FLP problems. The class of classical LP problems can be embedded into the class of FLP ones, moreover, for FLP problems we define the concept of duality and prove the weak and strong duality theorems—generalizations of the classical ones. The results are compared to the existing literature, see [8,9,13]. To illustrate the introduced concepts and results we present and discuss a simple numerical example.

2. Fuzzy relations and their properties

Let X be a nonempty topological space. By $\mathcal{F}(X)$ we denote the set of all fuzzy subsets A of X, where every fuzzy subset A of X is uniquely determined by the membership function $\mu_A : X \to [0, 1]$, and $[0, 1] \subset \mathbf{R}$ is a unit interval, $\mathbf{R}$ is the Euclidean space of real numbers. We say that the fuzzy subset A is *crisp* if μ_A is a characteristic function of A, i.e. $\mu_A : X \to \{0, 1\}$. It is clear that the set of all subsets of X, $\mathcal{P}(X)$, can be isomorphically embedded into $\mathcal{F}(X)$.

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doi:10.1016/j.fss.2005.11.022

Let

$$[A]_\alpha = \{x \in X | \mu_A(x) \geqslant \alpha\} \quad \text{for } \alpha \in (0, 1],$$
$$[A]_0 = cl\{x \in X | \mu_A(x) > 0\},$$

where clB means a topological closure of B, $B \subset X$. For $\alpha \in [0, 1]$, $[A]_\alpha$ are called *α-cuts*. $[A]_0$ is usually called a *support of* A. A fuzzy subset A of X is *closed*, *bounded*, *compact* or *convex*, if $[A]_\alpha$ are closed, bounded, compact or convex subsets of X for every $\alpha \in [0, 1]$, respectively. By the *strict α-cut* we denote

$$(A)_\alpha = \{x \in X | \mu_A(x) > \alpha\} \quad \text{for } \alpha \in [0, 1),$$
$$(A)_1 = \bigcap_{\alpha \in [0,1)} (A)_\alpha.$$

Moreover, A is said to be *normal* if $[A]_1$ is nonempty. It is a well known fact that a fuzzy subset A of X is convex if and only if its membership function μ_A is quasiconcave on X, see e.g. [9] and also Definition 5. The *complement* of A, $\mathcal{C}A$, is defined by

$$\mu_{\mathcal{C}A}(x) = 1 - \mu_A(x) \tag{1}$$

for all $x \in X$.

In the set theory, a *binary relation P on X* is a subset of the Cartesian product $X \times X$, that is, $P \subset X \times X$.

Here, a *valued relation P on X* is a fuzzy subset of $X \times X$.

Evidently, any binary relation P *on* X can be isomorphically embedded into the class of valued relations on X by its characteristic function (i.e. membership function) μ_P. In this sense, any binary relation is valued.

The following convention is useful: the element $x \in X$ is considered as the fuzzy subset of X with the characteristic function μ_x as its membership function. In this way we obtain the isomorphic embedding of X into $\mathcal{F}(X)$, in this sense we write $X \subset \mathcal{F}(X)$. Now, we define a crucial concept of this paper—fuzzy relation.

Definition 1. A fuzzy subset $\tilde{P}$ of $\mathcal{F}(X) \times \mathcal{F}(X)$ is called a *fuzzy relation on* X, i.e. $\tilde{P} \in \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(X))$.

Definition 2. Let P be a valued relation on X. A fuzzy relation $\tilde{Q}$ on X is called a *fuzzy extension of relation P*, if for each $x, y \in X$, it holds

$$\mu_{\tilde{Q}}(x, y) = \mu_P(x, y). \tag{2}$$

A fuzzy relations on X will be denoted by the tilde, e.g. $\tilde{P}$.

Definition 3. Let $\Phi, \Psi : \mathcal{F}(X \times X) \to \mathcal{F}(\mathcal{F}(X) \times \mathcal{F}(X))$ be two mappings assigning to any valued relation a fuzzy relation and let $\mathcal{F}_0$ be a nonempty subset of $\mathcal{F}(X \times X)$. We say that the *mapping Φ is dual to Ψ* on $\mathcal{F}_0$, if

$$\Phi(\mathcal{C}P) = \mathcal{C}\Psi(P) \tag{3}$$

holds for all valued relations $P \in \mathcal{F}_0$. For Φ dual to Ψ on $\mathcal{F}_0$, $P \in \mathcal{F}_0$, the fuzzy relation $\Phi(P)$ is called *dual* to fuzzy relation $\Psi(P)$.

As a simple consequence of the above definition and the identity $\mathcal{C}\mathcal{C}P = P$, a mapping Φ is dual to Ψ on $\mathcal{F}_0$, if and only if the mapping Ψ is dual to Φ on $\mathcal{F}_0$. The analogical statement holds for the dual fuzzy relations $\Phi(P)$ and $\Psi(P)$.

From now on, throughout this paper we shall consider $X = \mathbf{R}^n$, where $\mathbf{R}^n$ is the n-dimensional Euclidean space, particularly $X = \mathbf{R}^1 = \mathbf{R}$. In the following definition we first present possibility and necessity indices introduced originally in [2] and then define a suitable class of fuzzy numbers called here fuzzy quantities. Then, we shall derive some basic properties of this class.

Definition 4. Let A, B be fuzzy sets with the membership functions $\mu_A : \mathbf{R} \to [0, 1]$, $\mu_B : \mathbf{R} \to [0, 1]$, respectively. Let

$$Pos(A \preccurlyeq B) = \sup\{\min(\mu_A(x), \mu_B(y)) | x \leqslant y, x, y \in \mathbf{R}\}, \tag{4}$$