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Multiparametric Linear Programming with Applications to Control

C.N. Jones*, M. Barić and M. Morari

Automatic Control laboratory, ETH Zurich, Physikstrasse 3, Zurich, Switzerland

Parametric programming has received a lot of attention in the control literature in the past few years because model predictive controllers (MPC) can be posed in a parametric framework and hence pre-solved offline, resulting in a significant decrease in on-line computation effort. In this paper we survey recent work on parametric linear programming (pLP) from the point of view of the control engineer. We identify three types of algorithms, two arising from standard convex hull paradigms and one from a geometric intuition, and classify all currently proposed methods under these headings. Through this classification, we identify a third standard convex hull approach that offers significant potential for approximation of pLPs for the purpose of control. We present the resulting algorithm, based on the beneath/beyond paradigm, that computes low-complexity approximate controllers that guarantee stability and feasibility.

Keywords: Parametric linear programming; approximation; explicit model predictive control

1. Introduction

It is standard practice to implement a model predictive controller (MPC) by solving an optimization problem on-line. For example, when the system is linear, the constraints are polyhedral and the cost is linear or quadratic, this amounts to computing a single linear or quadratic program (LP/QP) at each sampling instant. In recent years, it has become well-known that for this class of systems the optimal input is a piecewise affine function (PWA) defined over a polyhedral partition of the feasible states [10,34,59]. By pre-computing this PWA function off-line, the on-line calculation of the control input then becomes one of evaluating the PWA function at the current measured state, which allows for significant improvements in sampling speed.

In this paper, we restrict our attention to the case when the cost is linear, 1- or ∞-norm. The computation of the optimal PWA function, mapping the measured state to the control input, can then be posed as the following (multi) parametric linear program (pLP) with parameters entering in the right-hand side (RHS) of the constraints:

$$\min_{y}\{b^T y | (\theta, y) \in \mathcal{P}\}, \tag{1}$$

where $\theta \in \mathbb{R}^d$ is the parameter, or state, $y \in \mathbb{R}^m$ is the optimizer, or control input and slack variables and $\mathcal{P}$ is a polyhedron, which incorporates the system constraints and is assumed bounded.

Both the properties and the computation of multiparametric linear programs (pLP) has been discussed in the literature for many years. Early work on parametric linear programming dates back to W. Orchard-Hays in his Master's thesis of 1952, published in [50] and Saaty and Gass [54]. Further studies have been ongoing ever since, primarily focusing on one-dimensional parametric problems and sensitivity analysis. The interested reader is referred to [24] for a survey of early work. There has, however, been a significant resurgence of interest on multi-parametric linear programming in the past few years in the control community due to the aforementioned link with model predictive control.

**Correspondence to*: C.N. Jones, E-mail: cjones@ee.ethz.ch

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In this paper we will survey current methods of solving the pLP (1) from the point of view of control theory. We will first demonstrate that all parametric linear programs of the form (1) can be posed as vertex or facet enumeration problems and then conduct the survey by classifying the methods in the literature into one of the three main paradigms for vertex/facet enumeration known in computational geometry and a fourth paradigm arising from geometrical intuition. It will be seen that current methods fall into three of these four classes of algorithms.

We will then propose a new method based on the fourth, as yet unstudied paradigm, and demonstrate various beneficial properties of this approach to control. Specifically, we will see that methods based on the so-called beneath/beyond method can be used to generate approximate, yet stabilizing, and invariance-inducing control laws. Furthermore, tuning knobs are available that allow the designer to trade-off between complexity, region of attraction and performance of the approximate control law.

The remainder of the paper is organized as follows. Sections 2 and 3 provide the basic background and problem setup for parametric linear programming. Section 4 demonstrates that all pLP of the form (1) can be posed as vertex or facet enumeration problems. Section 5 presents a survey of current methods in terms of the established approaches to vertex/facet enumeration and their relative complexity is analyzed in Section 6. Finally, a numerical examples demonstrating effectiveness of the proposed beneath/beyond scheme is given in Section 7.

2. Notation and Background

If $A \in \mathbb{R}^{m\times n}$ and $I \subseteq \{1, \ldots, n\}$, then $A_{*,I} \in \mathbb{R}^{m\times|I|}$ is the matrix formed by the columns of A indexed by I. If $c \in \mathbb{R}^n$ is a vector then c_I is the vector formed by the elements of c in I. If $R \subseteq \{1, \ldots, m\}$ then we will use the notation $A_{R,*} \in \mathbb{R}^{|R|\times n}$ to denote the matrix formed by the rows of A indexed by R.

A set S is called *affine* if $(1-\lambda)x + \lambda y \in S$ for all $x,\ y \in S$ and $\lambda \in \mathbb{R}$ and *convex* if λ is restricted to lie between zero and one. The *affine (convex) hull* of a set S is the intersection of all affine (convex) sets containing S, denoted aff S (conv S). If S is a finite point set, then $\text{aff}\, S = \{\sum s_i\lambda_i | s_i \in S, \sum\lambda_i = 1\}$ and $\text{conv}\, S = \{\sum s_i\lambda_i | s_i \in S, \sum\lambda_i = 1, \lambda_i \geq 0\}$. The *dimension* of a set is the dimension of the subspace parallel to its affine hull.

A *polyhedron* is the intersection of a finite number of halfspaces and a *polytope* is a bounded polyhedron. If $P = \{x | Ax \leq b\}$ is a polyhedron and $H = \{x | a^T x \leq d\}$ is a halfspace such that $P \subseteq H$, then $P \cap \{x | a^T x = d\}$ is a *face* of P. One- and zero-dimensional faces are called *edges* and *vertices* respectively. If P is of dimension d, then (d-1)- and (d-2)-dimensional faces are called *facets* and *ridges* respectively. The inequality $A_{i,*}x \leq b_i$ is called *redundant* if $P = \{x | A_{\{1,\ldots,n\}\setminus\{i\},*}x \leq b_{\{1,\ldots,n\}\setminus\{i\}}\}$ and *irredundant* otherwise.

A set C is called a cone if for every $x \in C$ and scalar $\alpha \geq 0$, we have $\alpha x \in C$. The columns of a matrix $F \in \mathbb{R}^{m\times n}$ are called the *generators* of the cone $C = \textit{cone}\ F := \{F\alpha \mid \alpha \geq 0\}$, if F is a single column, then cone F is called a *ray*. The generator $A_{*,i}$ is called *redundant* if $F_{*,i} \in \text{cone}(\text{F}_{*.\{1,\ldots,n\}\setminus\{i\}})$ and *irredundant*, or *extreme* otherwise.

The *Minkowski sum* of two sets, denoted $A \oplus B$ is defined as $A \oplus B := \{x + y\ | x \in A,\ y \in B.\}$. Every polyhedron P can be written as the Minkowski sum of a convex hull of a finite point set V and the *conic hull* of a finite number of rays R, $P = \text{conv}\ V \oplus \text{cone}R$.

A hyperplane $h := \{x | a^T x = c\}$ is said to *separate* two sets S_1 and S_2 if $S_1 \subseteq \{x | a^T x \leq c\}$ and $S_2 \subset \{x | a^T x \geq c\}$.

3. Preliminaries

3.1. Optimal Control

The recent interest in parametric programming in the control community has arisen from the ability to pose certain optimal control problems as parametric problems and thereby pre-compute the optimal control law offline. In this paper, we are specifically interested in the following standard semi-infinite horizon optimal control problem:

$$J^*(x) = \min_{\{u_0,\ldots,u_{N-1}\}} V_N(x_N) + \sum_{i=0}^{N-1} l(x_i, u_i) \qquad (2)$$

$$\begin{aligned} \text{s.t. } x_{i+1} &= Ax_i + Bu_i, \quad \forall i = 0, \ldots, N-1 \\ (x_i, u_i) &\in \mathcal{X} \times \mathcal{U}, \quad \forall i = 0, \ldots, N-1 \\ x_N &\in \mathcal{X}_F, \\ x_0 &= x \end{aligned}$$

where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_F$ are polytopic constraints on the states and inputs and the stage cost l is defined as $l(x_i, u_i) := \|Qx_i\|_p + \|Ru_i\|_p$. Under the standard assumptions that $\mathcal{X}_\mathcal{F} \subseteq \mathcal{X}$ is an invariant set, V_N is a Lyapunov function and that the decay rate of V_N is greater than the stage cost within the set $\mathcal{X}_F$ then the problem (2) generates a stabilizing control law when applied in a receding horizon fashion [44]. If the norm