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Optimal and strongly optimal solutions for linear programming models with variable parameters

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Abstract

This paper deals with linear programming (LP) models with variable parameters and introduces two concepts for this class of problems: optimal solution and strongly optimal solution. Also, it seeks necessary and sufficient conditions for a feasible solution to be optimal or strongly optimal.

Keywords: Linear programming; Optimality; Pareto optimality; KKT conditions; Representation theorem

1. Introduction

In the real applications of the field of optimisation, there are linear programming (LP) problems whose parameters (i.e., technological coefficients, requirement values, and cost coefficients) are not precisely fixed and can vary within some prescribed intervals (see [2,4,5] as well as Section 5 of the present paper). In this paper, we deal with this class of LP models, which is a more general class compared to the one discussed in [5], and introduce two key concepts: optimal solution and strongly optimal solution. The main aim of this paper is to find the necessary and sufficient conditions for a feasible solution to be optimal or strongly optimal.

The rest of this paper is organized as follows: In Section 2, LP models with variable parameters are defined, and the two concepts of optimal solution and strongly optimal solution are introduced. Section 3 and Section 4 contain necessary and sufficient conditions for optimality and strong optimality, respectively. Section 5 addresses some applications of the results of this paper.

2. LP models with variable data

Let us consider the following problem

$$\begin{aligned} &\min \mathbf{cx} \\ &\text{s.t.}\quad \mathbf{Ax} = \mathbf{b} \quad \mathbf{x} \in P \subset \mathbb{R}^n, \end{aligned} \tag{1}$$

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where **c** and **b** are two $1 \times n$ and $m \times 1$ vectors, respectively; **A** is an $m \times n$ matrix, and P is any subset of $\mathbb{R}^n$. The entries of **A**, **b**, and **c** are not fixed, but they can assume any value within some prescribed intervals, i.e.,

$$a_{ij}^- \leqslant a_{ij} \leqslant a_{ij}^+,$$
$$b_i^- \leqslant b_i \leqslant b_i^+,$$
$$c_j^- \leqslant c_j \leqslant c_j^+.$$

Now we define the set of feasible solutions for (1) as follows:

$$X := \bigcup_{\substack{\mathbf{A}^- \leqslant \mathbf{A} \leqslant \mathbf{A}^+ \\ \mathbf{b}^- \leqslant \mathbf{b} \leqslant \mathbf{b}^+}} \{\mathbf{x} \in P : \mathbf{A}\mathbf{x} = \mathbf{b}\},$$

where $\mathbf{A}^- = [a_{ij}^-]_{m\times n}$, $\mathbf{A}^+ = [a_{ij}^+]_{m\times n}$, $\mathbf{b}^- = [b_i^-]_{m\times 1}$, and $\mathbf{b}^+ = [b_i^+]_{m\times 1}$. Also, the comparison of matrices is componentwise. The following lemma is useful for later purposes.

Lemma 1. *Suppose that*

$$\mathbf{D}^- = \begin{pmatrix} \mathbf{A}^- & -\mathbf{b}^+ \\ \mathbf{0}_{1\times n} & 1 \end{pmatrix}, \qquad \mathbf{D}^+ = \begin{pmatrix} \mathbf{A}^+ & -\mathbf{b}^- \\ \mathbf{0}_{1\times n} & 1 \end{pmatrix}, \tag{2}$$

$$\mathcal{D} = \{\mathbf{D} : \mathbf{D}^- \leqslant \mathbf{D} \leqslant \mathbf{D}^+\},$$

$$Z^* = \bigcup_{\mathbf{D}\in\mathcal{D}} \left\{ \mathbf{z} = \begin{pmatrix} \mathbf{x} \\ y \end{pmatrix} \in P \times \mathbb{R}_+ : \mathbf{D}\mathbf{z} = \begin{pmatrix} \mathbf{0}_{m\times 1} \\ 1 \end{pmatrix} \right\},$$

and

$$Z^{**} = \left\{ \mathbf{x} \in P : \exists y \in \mathbb{R}_+ ; \begin{pmatrix} \mathbf{x} \\ y \end{pmatrix} \in Z^* \right\},$$

then $X = Z^{**}$.

Proof. If $x \in X$, then there exist **A** and **b** such that

$$\mathbf{A}^- \leqslant \mathbf{A} \leqslant \mathbf{A}^+, \qquad \mathbf{b}^- \leqslant \mathbf{b} \leqslant \mathbf{b}^+, \quad \text{and} \quad \mathbf{A}\mathbf{x} = \mathbf{b}.$$

Defining

$$\mathbf{D} = \begin{pmatrix} \mathbf{A} & -\mathbf{b} \\ \mathbf{0}_{1\times n} & 1 \end{pmatrix},$$

it can simply be shown that $\mathbf{D} \in \mathcal{D}$. Furthermore,

$$\mathbf{D}\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & -\mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{x} - \mathbf{b} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

Therefore $\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \in Z^*$, and hence $\mathbf{x} \in Z^{**}$.

If $\mathbf{x} \in Z^{**}$, then there exists a matrix $\mathbf{D} = \begin{pmatrix} \mathbf{A} & -\mathbf{b} \\ \mathbf{0}_{1\times n} & 1 \end{pmatrix}$ and $y \in \mathbb{R}_+$ such that $\mathbf{D} \in \mathcal{D}$ and $\mathbf{D}\begin{pmatrix} \mathbf{x} \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$. These imply that

$$\mathbf{A}^- \leqslant \mathbf{A} \leqslant \mathbf{A}^+, \qquad \mathbf{b}^- \leqslant \mathbf{b} \leqslant \mathbf{b}^+, \qquad \mathbf{A}\mathbf{x} - \mathbf{b}y = \mathbf{0}, \quad \text{and} \quad y = 1.$$

Therefore, $x \in X$. ■

To continue, we define two concepts: *optimal solution* and *strongly optimal solution*, for (1) as follows:

Definition 1. $\bar{\mathbf{x}}(\mathbf{c})$ is called an optimal solution for (1) if it solves $\min_{\mathbf{x}\in X} \mathbf{c}\mathbf{x}$, for some $\mathbf{c}^- \leqslant \mathbf{c} \leqslant \mathbf{c}^+$.

Definition 2. $\bar{\mathbf{x}}$ is called a strongly optimal solution for (1) if it solves $\min_{\mathbf{x}\in X} \mathbf{c}\mathbf{x}$, for all $\mathbf{c}^- \leqslant \mathbf{c} \leqslant \mathbf{c}^+$.