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Lower bound shakedown analysis by using the element free Galerkin method and non-linear programming

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ABSTRACT

Shakedown analysis is a powerful tool for assessing the safety of structures under variable repeated loads. By using the element free Galerkin (EFG) method and non-linear programming, a novel numerical solution procedure is developed to perform lower bound shakedown analysis of structures made up of elasto-perfectly plastic material. The numerical implementation is very simple and convenient because it is only necessary to construct an array of nodes in the domain under consideration. The reduced-basis technique is adopted here to solve the mathematical programming iteratively in a sequence of reduced self-equilibrium stress subspaces with very low dimensions. The self-equilibrium stress field is expressed by linear combination of several self-equilibrium stress basis vectors with parameters to be determined. These self-equilibrium stress basis vectors are generated by performing an equilibrium iteration procedure during elasto-plastic incremental analysis. The Complex method is used to solve the non-linear programming and determine the lower bound of shakedown load. The proposed numerical method is verified by using several numerical examples and the results show good agreement with other available solutions.

1. Introduction

The design of engineering structures subjected to variable repeated loads demands a realistic assessment of the safety margin with respect to failure. A particular kind of failure in the case of variable repeated loads is caused by an unlimited accumulation of plastic strains during the loading process, leading to either incremental collapse (characterized by unbounded deformation growth for each cycle of loading) or alternating plasticity (eventually leading to fractures by a low cycle fatigue type phenomenon). If, on the contrary, after some time plastic strains cease to develop further and the structure responds purely elastically to the applied variable loads, the structure is said to shake down. The shakedown of a structure indicates that the damage stops evolving after a finite number of loading cycles. This is due to the fact that a stationary self-equilibrium stress field is formed and the total dissipated energy becomes stationary. Therefore, the prediction of shakedown or collapse of structures under variable repeated loads is very important and useful for structural design and safety assessment, and has attracted the attention of many researchers [1–25].

Some designers hope to solve such problems by the elasto-plastic incremental analysis. However, it necessitates greater calculation efforts and requires detailed loading histories, which often are unavailable or uncertain in engineering situations. Shakedown analysis, a generalization of limit analysis, is a useful alternative to step-by-step method, particularly when only the upper and lower limits of the loading histories are known. The primary merit of shakedown analysis is that it enables the computation of the shakedown load against failure without resorting to the time-stepping evolutive solutions. But on the other hand, shakedown analysis is faced with great difficulty in numerical computation. With solution procedures, it is mostly centered on mathematical programming [3,4]. This mathematical programming has excessive independent variables and constraint conditions after discretization, in general is a large scale non-linear programming, and hence is usually very difficult to be solved (i.e. the obstacle of high dimension). At present many scholars have made great efforts to develop efficient computational methods for shakedown analysis. For example, Casciaro and Garcea [21] proposed an incremental iterative method for defining shakedown boundaries of frame structures. This method is based on a path-following iterative scheme similar to that used in limit analysis. Garcea et al. [15] extended this incremental iterative method to the shakedown analysis of two-dimensional flat structures in both the cases of plane stress and plane strain. Ngo and Tin-Loi [22] proposed the *p*-adaptive finite element method (FEM) as a robust and accurate approach to perform shakedown analyses of two-dimensional plane strain problems. This method shows great promise because it can overcome incompressibility locking and does not need extensive meshing. Moreover, incorporation of adaptive scheme at both elastic analysis and yield surface linearization levels is particularly computationally attractive in

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increasing performance of the p-adaptive FEM for shakedown analysis. Some other efficient numerical methods for shakedown analysis have also been reported, such as those by Boulbibane and Ponter [17], Makrodimopoulos [18], Krabbenhøft [19], and so on. Here we concentrate on the reduced-basis technique [1–4], which enables us to evaluate efficiently the shakedown load directly from the static shakedown theorem.

Up to now, most numerical methods for shakedown analysis are by means of the mesh-based methods such as finite element method and boundary element method. As an important alternative approach to eliminate the well known drawbacks in the mesh-based methods, meshless method has received much attention in recent years, due to their flexibility, and, most importantly, due to their potential in negating the need for the human-labor intensive process of constructing geometric meshes. Some representative examples are the element free Galerkin (EFG) method [26,27], the meshless local Petrov–Galerkin (MLPG) method [28,29], the reproducing kernel particle method (RKPM) [30], the smooth particle hydrodynamics (SPH) [31] and so on. The EFG method adopted here is a Galerkin discretization technique with the help of shape functions constructed using the moving least squares (MLS) approximation. To use the MLS approximation, it is only necessary to construct an array of nodes in the domain under consideration. Because no element connectivities are needed, the numerical procedure of the EFG method is quite simple. The EFG method also requires no post-processing for the output of strains and stresses or other field variables which are derivatives of the primary-dependent variables since these quantities are already very smooth. Meanwhile, the EFG method can avoid volumetric locking for nearly incompressible materials [26,27]. Furthermore, the computational results from the EFG method are of higher accuracy [26,27]. Consequently, the EFG method is particularly suitable for lower bound shakedown analysis due to the fact that it can accurately compute the fictitious elastic stress field and the self-equilibrium stress field without resorting to meshes or elements. The above advantages are so attractive that applying the EFG method to numerical shakedown analysis is of great interest and deserves study.

In this paper, our attention is focused on the EFG solution procedure for lower bound shakedown analysis. The considered structure is made up of isotropic, elasto-perfectly plastic material governed by von Mises' yield condition and Drucker's postulate. Based on the static shakedown theorem, shakedown analysis is transformed into a problem of mathematical programming whose optimization variables are the self-equilibrium stress field and the shakedown load factor. The domain discretization is based on the EFG method, where the MLS approximation is utilized to construct trial functions and the penalty method is used to impose the essential boundary conditions. The reduced-basis technique is adopted to express the self-equilibrium stress field by linear combination of several self-equilibrium stress basis vectors with parameters to be determined. These self-equilibrium stress basis vectors are generated by performing an equilibrium iteration procedure of elasto-plastic incremental analysis. The resulting optimization formulation for shakedown analysis is reduced to a non-linear programming with the inequality constraints of yield conditions at every Gaussian point for all corners of the load domain. Its solution can be obtained effectively by the complex method. Implementation details and numerical examples are presented to demonstrate the effectiveness of the developed method.

2. Static theorem of shakedown analysis

The lower bound of shakedown load of an elasto-perfectly plastic structure can be obtained using the static theorem of shakedown analysis. The static shakedown theorem can be stated as follows [14]: a structure will shake down to the prescribed loading range if there exists a time independent self-equilibrium stress field $\rho_{ij}(\boldsymbol{x})$ which, superimposed on the fictitious elastic stress field $\sigma_{ij}^{\mathrm{E}}(x,t)$, yields the total stress $\sigma_{ij}(\boldsymbol{x},t)$ not violating the yield condition at any point of the structure and for all possible load combinations, namely:

$$\varphi[\sigma_{ij}(\boldsymbol{x},t)] = \varphi[\sigma_{ij}^{\mathrm{E}}(\boldsymbol{x},t) + \rho_{ij}(\boldsymbol{x})] \leqslant 0 \quad \forall \boldsymbol{x} \in \Omega. \tag{1}$$

Here, $\varphi(\cdot)$ is the yield function, $\sigma_{ij}(\boldsymbol{x},t)$ is the actual stress due to the variable repeated loads $\boldsymbol{p}(\boldsymbol{x},t), \sigma_{ij}^{\mathrm{E}}(x,t)$ denotes the fictitious elastic stress that would appear had the structure responded to the applied loads in a purely elastic manner, and $\rho_{ij}(\boldsymbol{x})$ represents a self-equilibrium stress field that must satisfy equilibrium requirements within the body Ω and vanish on the part Γ_t of the surface where tractions are prescribed:

$$\rho_{ij,j} = 0 \quad \text{in} \quad \Omega, \tag{2a}$$

$$\rho_{ij} n_j = 0 \quad \text{on} \quad \Gamma_t, \tag{2b}$$

where n_j is the unit outward normal to the boundary Γ_t. It should be noticed that the classical conditions are assumed here, namely, small displacement gradients and hence linear kinematic relations, quasistatic loading, and an elasto-perfectly plastic material that is stable in Drucker's Sense.

By means of the static shakedown theorem, lower bound shakedown analysis can now be formulated as the following non-linear programming problem:

$$\beta^s = \max \quad \beta \tag{3a}$$

$$\text{s.t.} \quad \varphi[\sigma_{ij}(\boldsymbol{x},t)] = \varphi[\beta\sigma_{ij}^{\mathrm{E}}(\boldsymbol{x},t) + \rho_{ij}(\boldsymbol{x})] \leqslant 0 \quad \forall \boldsymbol{x} \in \Omega, \tag{3b}$$

$$\rho_{ij,j}(\boldsymbol{x}) = 0 \quad \forall \boldsymbol{x} \in \Omega, \tag{3c}$$

$$\rho_{ij}(\boldsymbol{x}) n_j = 0 \quad \forall \boldsymbol{x} \in \Gamma_t, \tag{3d}$$

where β is load factor. The above static formulation is of bounding character, which means that if we can find a time independent self-equilibrium stress field $\rho_{ij}(\boldsymbol{x})$ and a corresponding load factor β such that the yield condition (3b) is satisfied for all $\boldsymbol{x} \in \Omega$ and for all $t > 0$, then provides a lower bound to the actual shakedown load factor β^s.

3. Review of the element free Galerkin (EFG) method

In this paper, the element free Galerkin method (EFG) is employed to solve the formulation (3) of lower bound shakedown analysis. The EFG method is a Galerkin discretization technique based on the moving least squares (MLS) approximation. The first form of the EFG method was reported by Nayroles et al. [32]. Their method was called the diffuse element method (DEM). The EFG method developed by Belytschko et al. [26] includes certain terms in the derivatives of the MLS approximation that are omitted in the DEM. The EFG method is very attractive in many problems because of its good accuracy, ease in formulation and high stability. Up to now, remarkable successes of the EFG method have been achieved in solving plane elasticity [26,27], steady-state heat conduction [26], plate analysis [33], dynamic fracture mechanics [34] and so on.

3.1. The moving least squares (MLS) approximation

In general, meshless methods require a local interpolation or approximation to represent the trial function with the values (or the fictitious values) of the unknown variable at some randomly located nodes. The approximation in the EFG method is the moving least squares approximation in which the function $u(\boldsymbol{x})$ is approximated [26–29] by

$$u^h(\boldsymbol{x}) = \boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{x}) = \sum_{j=1}^{m} p_j(\boldsymbol{x}) a_j(\boldsymbol{x}), \tag{4}$$