International Journal of Mechanical Sciences 51 (2009) 89–104

Contents lists available at ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

A discontinuous quasi-upper bound limit analysis approach with sequential linear programming mesh adaptation

G. Milani [a], *, P.B. Lourenço [b]

[a] Department of Structural Engineering (IBK), Swiss Federal Institute of Technology (ETH), Wolfgang-Pauli-Strasse 15, 8093 Zürich, Switzerland
[b] Department of Civil Engineering, School of Engineering, University of Minho, Azurém, 4800-058 Guimarães, Portugal

ARTICLE INFO

Article history:
Received 17 January 2008
Received in revised form
17 October 2008
Accepted 24 October 2008
Available online 5 November 2008

Keywords:
Upper bound limit analysis
Rigid elements
Interfaces
Sequential linear programming
Associated flow rule
Frictionless and frictional materials

ABSTRACT

In this paper, a simple discontinuous upper bound limit analysis approach with sequential linear programming mesh adaptation is presented. Rigid, infinitely strong triangular elements with both linear and Bezier curved edges are considered. A possible jump of velocities is allowed at the interfaces between contiguous elements, thus allowing plastic dissipation on curved interfaces. Bezier curved edges are used with the sole aim of improving the element performance when dealing with limit analysis problems involving curved sliding lines. The model performs poorly for unstructured meshes (i.e. at the initial iteration), being unable to reproduce the typical plastic deformation concentration on inclined slip lines. Therefore, an iterative mesh adaptation based on sequential linear programming is proposed. A simple linearization of the non-linear constraints is performed, allowing to treat the non-linear programming (NLP) problem with consolidated linear programming (LP) routines. The choice of inequalities constraints on elements nodes coordinates turns out to be crucial on the algorithm convergence.

Several examples are treated, consisting in the determination of failure loads for ductile, purely cohesive and cohesive-frictional materials. The results obtained at the final iteration fit well, for all the cases analyzed, previously presented numerical approaches and analytical predictions.

1. Introduction

The determination of the ultimate load-bearing capacity of engineering structures deserves great consideration from a technical point of view. Despite the considerable research efforts done in the past decades in the field of finite element (FE) limit analysis, the linear elastic approach is generally considered as the most practical tool to obtain quantitative information for design purposes. Unfortunately, such analysis fails to give an idea of the structural behavior beyond the beginning of cracking. Especially for cohesive-frictional materials and in the case of masonry, this represents a non-negligible drawback. In fact, due to the relatively low tensile strength of such kind of materials, linear elastic analyses are unable to represent adequately the structural behavior, even in presence of very low load levels. For this reason, limit analysis is a promising alternative, giving the possibility to predict failure loads and failure mechanisms with a moderate computational effort, requiring only a few mechanical parameters at failure for the simulations. In geomechanics, limit analysis provides a useful method for assessing the load-bearing capacity of structures (e.g. footings, retaining walls, etc.) and the stability of slopes and excavations.

While for linear elastic analyses robust and efficient factorization routines are at disposal in FE solvers, for limit analysis only linear programming (LP) routines able to tackle problems involving several variables are needed. Nowadays, commercial LP packages can compete favourably both for stability and time required for the simulations with elastic FE solvers. Furthermore, the time requirements to construct the FE model are the same as for the elastic analysis.

Nonetheless, limit analysis combines, on one hand, sufficient insight into collapse mechanisms, ultimate stress distributions—at least in critical sections and load capacities, and on the other hand, simplicity to be cast into a practical computational tool. Given the difficulties in obtaining reliable experimental data for frictional materials, another appealing feature of limit analysis is the reduced number of necessary material parameters.

Several efforts have been made in this field in the last decades by many authors (see for instance Refs. [1–7]), with the aim of solving the linear optimization problem by means of non-linear programming routines (NLP), usually avoiding to perform a linearization of the material strength domain. This allowed

* Corresponding author. Formerly at: Department of Engineering, University of Ferrara, Via Saragat 1 44100 Ferrara, Italy. Tel.: +39 0532 974911; fax: +39 0532 974870.
E-mail addresses: milani@ibk.baug.ethz.ch, gmilani@ing.unife.it (G. Milani).

0020-7403/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijmecsci.2008.10.010

a further improvement in the numerical efficiency of FE limit analysis programs.

Another important aspect of the FE approach within limit analysis is that the classical lower and upper bound theorems allow to rigorously bounding the exact limit load for a perfectly plastic structure. When the bound theorems are implemented numerically in combination with the FE method, the ability to obtain tight bracketing depends not only on the efficient solution of the arising optimization problem, but also on the effectiveness of the elements employed. Elements for (strict) upper bound analysis pose a particular difficulty, since the flow rule is required to hold throughout each element, yet it can only be enforced at a finite number of points. The standard choice for this type of analysis has been the constant strain element combined with discontinuities in the displacement field (see for instance Sloan and Kleeman [1]).

Nevertheless, the accuracy of the approach is highly dependent on the alignment of the discontinuities, meaning that it can perform poorly if an unstructured mesh is employed. Such a poor performance increases when dealing with rigid, infinitely strong elements, as is the case treated here. Plastic dissipation, in this case, occurs only at the interfaces between contiguous elements, thus constraining the collapse loads to be drastically dependent on the disposition of the interfaces in the mesh. In order to circumvent this limitation, re-meshing strategies could be adopted, as suggested in Refs. [8,9]. In this case, an iterative procedure with increasing number of optimization variables at successive iterations is needed. As an alternative, adaptive upper bound methods with linear elements and possible plastic deformation in both triangles and discontinuities, as proposed in Ref. [10], should be used.

Considering the drawbacks related to a triangular discretization, it also seems appealing the generalization (using adaptive schemes at both element and material level) of totally different procedures presented in the recent past in the literature, as for instance, the free Galerkin approach (see Ref. [11]) or the p-FEM [12].

Differently from existing algorithms available, the basic idea of the procedure proposed here consists in (a) limiting, as much as possible, optimization variables in order to make the numerical model fast and efficient and (b) reproducing general failure mechanisms involving curved discontinuities, utilizing few elements without dissipation in continuum (and therefore avoiding the introduction of additional plastic multipliers in continuum). Such requirements are somewhat contradictory, since it is well known that linear rigid elements perform well in the reproduction of complex curved collapse mechanisms, only if the mesh utilized is sufficiently refined (i.e. if curved edges, active in the dissipation process, are well approximated by segments).

In this framework, in order to comply with accuracy and limited computational effort both a linear and a curved Bezier triangular rigid element with possible dissipation along (curved) interfaces between adjoining elements are presented. Since dissipation can occur only at the interfaces between contiguous elements, a mesh adaptation algorithm able to enforce the shape of the interfaces to coincide with the actual slip lines is adopted. There are many reasons which justify the use of adaptive rigid elements with linear or curved edges. Among the others, the most important ones (which make, indeed, preferable the use of curved elements in some cases) are their ability to reproduce complex failure mechanisms with plastic dissipation concentrated on curved slip lines and the simplicity of the algorithm itself, consisting in the recursive utilization of a LP routine.

It is worth noting that the utilization of (a) curved elements, (b) relatively coarse meshes and (c) iterative LP schemes differs significantly from adaptive techniques recently presented in the technical literature (see for instance Refs. [10,13]), which are usually based on the utilization of NLP codes. The approach here presented, in fact, follows a classical procedure (attempted at the early stages of FE limit analysis research) based on LP and linearization of the failure surfaces. Furthermore, it concentrates exclusively on geometrical aspects (linearization of non-linear constraints and utilization of splines) rather than on numerical issues related to NLP and therefore may be of interest also for practical engineers not involved in the most recent optimization research.

Bezier curves are treated in the model in the same manner as linear interfaces, with the only difference that more plastic multiplier rates are required for each interface and that plastic dissipation is obtained resorting to numerical integration methods.

For both linear and curved elements, a simple linearization of the non-linear constraints is performed, allowing to treat the NLP problem with consolidated LP routines. The choice of inequalities constraints on elements nodes coordinates turns out to be crucial on the algorithm convergence.

Several meaningful examples are treated to validate the procedure proposed, consisting in the determination of failure loads of ductile (plate with central hole), purely cohesive and cohesive-frictional materials (indentation problems, masonry shear walls). The results obtained at the final iteration fit well, for all the cases analyzed, previously presented numerical approaches and, where available, analytical predictions.

2. Triangular upper bound limit analysis

In this section, an adaptive upper bound limit analysis conducted by means of triangular elements with both linear and curved Bezier edges and possible dissipation between adjoining elements is presented.

Linear elements can be regarded as a particular sub-class of curved elements. Anyway, the reasons at the base of the utilization of adaptive rigid elements with generally curved edges are (1) the capability of the method to reproduce complex failure mechanisms with non-linear slip lines more accurately, when compared with standard linear triangles and (2) the simplicity of the algorithm, consisting on a trivial recursive utilization of robust LP routines. For both linear and curved elements, it is worth noting that the method competes favourably with more classical re-meshing techniques [8,9]. In fact, while for the present analyses, a number of LP problems with the same number of optimization variables have to be solved to converge to suitable solutions, in re-meshing [8] the number of optimization variables increases at successive iterations, meaning that time required to perform the simulation becomes great near the optimal mesh.

2.1. The linear and the rigid splines element

Let us consider a triangular element T with curved edges, as shown in Fig. 1. Each edge of the element is constituted by cubic Bézier splines defined by four control points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$, see Fig. 1a. As well known from the technical literature [14–16], Cartesian coordinates of each point of the Bezier curve can be obtained in parametric form having at disposal $\mathbf{P}_1, \ldots, \mathbf{P}_4$ as follows:

$$\mathbf{P}(t) = -(t-1)^3\mathbf{P}_1 + 3t(t-1)^2\mathbf{P}_3 - 3t^2(t-1)\mathbf{P}_4 + t^3\mathbf{P}_2 \quad t \in [0,\ 1] \tag{1}$$

In the model, both a C0 and C1 continuity of the edges between adjoining elements can be required. In particular, when a C1