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# A linear programming approach to test efficiency in multi-objective linear fractional programming problems

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ABSTRACT

In a multi-objective linear fractional programming problem (MOLFPP), it is often useful to check the efficiency of a given feasible solution, and if the solution is efficient, it is useful to check strong or weak efficiency. In this paper, by applying a geometrical interpretation, a linear programming approach is achieved to test weak efficiency. Also, in order to test strong efficiency for a given weakly efficient point, a linear programming approach is constructed.



## 1. Introduction

Fractional programming (FP) deals with situations in which the ratio between two functions is considered; for example, cost/time, cost/volume, cost/profit, or other equations that measure the efficiency of a system. Sometimes, the optimization of the ratio of functions gives more insights into the situation than optimizing each function separately; see [1] for more details.

If the numerator and denominator in the objective function as well as the constraints are linear, we have a linear fractional programming problem (LFPP) as follows:

$$\begin{aligned} &\text{Max} \quad \frac{cx+\alpha}{dx+\beta}, \\ &\text{s.t.} \quad x \in S = \{x | Ax \leqslant b, x \geqslant 0\}, \end{aligned} \tag{1}$$

where $A$ is a real $m \times n$ matrix, $b \in R^m$, $x \in R^n$ and $S$ is a nonempty and bounded set. Charnes and Cooper [2] showed that if the denominator is constant in sign on the feasible region, the LFPP can be optimized by solving a linear programming problem. Schaible [3], Dinkelbach [4], Gilmore and Gomory [5], Craven [1], and others presented methods for solving FPs and LFPPs.

However, in many applications, there are two or more conflicting objective functions which are relevant, and some compromise must be sought between them. For example, a management problem may require the profit/cost, quality/cost, and

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other ratios to be maximized and these conflict. Such types of problems are inherently multi-objective linear fractional programming problems and can be written as

$$\begin{aligned} &\text{Max} \quad f_k(x) = \frac{c_k x + \alpha_k}{d_k x + \beta_k}, \quad k = 1, \ldots, K, \\ &\text{s.t.} \quad x \in S = \{x | Ax \leqslant b, x \geqslant 0\}, \end{aligned} \tag{2}$$

where $A$, $b$ and $x$ are as defined in problem (1), and $\forall x \in S$, $d_k x + \beta_k > 0$ ($k = 1, \ldots, K$). The feasible solution $x_1$ is $a(n)$ (strongly) efficient point iff there does not exist another feasible solution $x_2$ such that $f_k(x_2) \geqslant f_k(x_1)$ ($k = 1, \ldots, K$) and $f_k(x_2) > f_k(x_1)$ for some $k$. A feasible solution $x_1$ is a weakly efficient point iff there does not exist another feasible solution $x_2$ such that $f_k(x_2) > f_k(x_1)$ ($k = 1, \ldots, K$). The set of all strongly efficient points is denoted by $E$ and the set of all weakly efficient points is denoted by $E_w$.

There are several methodologies [1,6–8] to solve MOLFPPs. Kornbluth and Steuer [6,7] presented two different approaches to MOLFPP. Choo and Atkins [8] gave an analysis of the bicriteria linear fractional programming problem. To early of 1990, most of the methodologies for solving MOLFPPs were computationally burdensome. The application of fuzzy set theory was to overcome this difficulty [9]. Using fuzzy set theory, Luhandjula [10], Pal et al. [11], and Sakawa and Yano [12] presented different methods to solve and analyze MOLFPPs.

Metev and Gueorguieva [13] presented a nonlinear programming problem for finding the weakly efficient points, and a nonlinear programming problem to test weak efficiency in MOLFPPs. In this paper, using a simple geometrical interpretation, we propose a linear programming approach for testing strong efficiency and weak efficiency in MOLFPPs.

The paper is organized as follows. In Section 2, by a geometrical interpretation, we present a linear programming problem to test optimality in LFPP and a linear programming problem to test week efficiency in MOLFPP and a linear programming problem to test strong efficiency in MOLFPP and some facts about these linear programs is stated. At last, we present the remarks and conclusion.

## 2. Methodology

Given $(m, n) \in R^2$, $m > 0$, $n > 0$, assume that $y = \frac{n}{m}x$ is a straight line that passes through $(m, n)$ and the origin. Point $(m', n') \in R^2$, with $m' > 0$, $n' > 0$, is above the line $y = \frac{n}{m}x$ iff the gradient of the line passing through $(m', n')$ and the origin is greater than $\frac{n}{m}$, that is $\frac{n'}{m'} > \frac{n}{m}$. This is shown in Fig. 1.

**Theorem 1.** *If $m$, $n$, $m'$, $n'$ are positive real numbers, then:*

$$\frac{n'}{m'} > \frac{n}{m} \iff \exists\,\theta\,\exists\,d^-\,\exists\,d^+\left(\theta \in R^+, d^-, d^+ \in R^{\geqslant 0}, n' - d^+ = n\theta, m' + d^- = m\theta, d^- + d^+ > 0\right).$$

**Proof.**

$$\frac{n'}{m'} > \frac{n}{m} \iff \frac{n'}{n} > \frac{m'}{m} \iff \exists\,\theta\left(\theta \in R^+, \frac{n'}{n} > \theta > \frac{m'}{m}\right) \iff \exists\,\theta(\theta \in R^+, n' > n\theta, m' < m\theta)$$
$$\iff \exists\,\theta\,\exists\,d^-\,\exists\,d^+\left(\theta \in R^+, d^-, d^+ \in R^{\geqslant 0}, n' - d^+ = n\theta, m' + d^- = m\theta, d^- + d^+ > 0\right). \qquad \square$$

So $\frac{n'}{m'} > \frac{n}{m}$ is equivalent to the statement that can move along direction $(m, n)$ with step length $\theta > 0$ to reach $(m\theta, n\theta)$, thence along $(-1, 0)$ with step length $d^-$ to reach $(m\theta - d^-, n\theta)$, and from that point along $(0, 1)$ with step length $d^+$ to reach $(m\theta - d^-, n\theta + d^+) = (m', n')$, which is shown in Fig. 2.

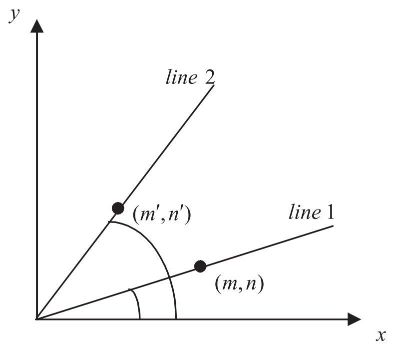


**Fig. 1.** The gradient of line 2 is greater than that of line 1, and so $(m', n')$ is above line 1.