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Allocation of the EU Parliament seats via integer linear programming and revised quotas

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ABSTRACT

We deal with the problem of assigning seats to the European Parliament within the special requirements imposed by the rules of the EU. Since the usual rounding techniques, like in the divisor methods, may fail to satisfy these requirements, we propose to use integer linear programming (ILP) to provide at the same time rounding and satisfaction of the requirements. Using ILP makes central the choice of quotas to which the seats should be as close as possible. We investigate how the special requirements can affect the very definition of quotas, and define projective quotas. Finally we compare the various methods by using the EU Parliament data.

1. Introduction

This note contains some reflections after reading the document Grimmett et al. (2011), also called 'The Cambridge Compromise'. The allocation of seats to the constituencies of a state is a well studied problem (see for instance Balinski and Young, 2001) and seemingly there is little else to say. All methods are designed to satisfy the following obvious requirement:

1. a constituency, i.e., a country for the European Union, must not receive less seats than a smaller country.

However, the rules for the EU Parliament introduce three new concepts which alter the usual framework (see Lamassoure and Severin, 2007):

2. no country can receive more seats than a stated upper bound;
3. no country can receive less seats than a stated lower bound;
4. seats must satisfy the so called 'degressive proportionality' requirement.

Degressive proportionality means that the ratio of population/seats should be an increasing function of the population. In other words, in a larger country more people are needed to form a seat.

Finding a simple method of assigning seats within these rules is the subject of Grimmett et al. (2011). The proposal is to first assign to each country a number of seats equal to the lower bound, and then to assign the remaining seats via a divisor method, by possibly capping the seats whenever they exceed the upper bound.

As shown in Grimmett et al. (2011) the method may fail to satisfy the degressive proportionality rule. With regard to this point the authors suggest to weaken the rule by requiring degressive proportionality only before rounding. As a matter of fact it seems difficult to reconcile rounding schemes, like for instance the ones used in divisor methods or in largest remainders methods, with degressive proportionality.

Here we take a different attitude in order to comply with these rules. As they are, they seem particularly suited to model the problem via an integer linear programming (ILP) problem. The rules become hard constraints that must be satisfied by the seat assignment. In addition an objective function has to be added to the problem to get proportionality (in a way still to be made more precise) as much as possible.

It may be argued that using ILP is not transparent. A specific mathematical knowledge is required in order to model the problem and to solve it. However, in our opinion, a divisor method also requires some mathematical skills and is not amenable to the layman. Nowadays, linear programming packages are largely available (even on spreadsheets) and the model, given its modest size and simple structure, can be easily replicated in most governmental offices and university departments, so that educated people can check the result without any particular effort.

This statement may look in contrast to the attitude taken in Simeone and Serafini (in press-b) where verifiability for the layman of a biproportional seat assignment is pursued, with seats assigned according to the method suggested in Simeone and Serafini (in press-a). We think that assigning seats to countries is indeed a delicate issue, but more at a governmental level, where tools to understand an approach and check a result are somehow available. On the contrary, assigning seats to parties on the basis of the votes expressed by the citizens requires some

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form of 'understanding' the method at the same level of the voters themselves.

The proportionality issue can be approached by defining rational numbers, so called 'quotas', to which the integer numbers representing the seats should adhere as much as possible, for instance minimizing some form of deviation from the quotas. This way the problem of assigning the seats is split into two separate problems, namely first defining the quotas to deal with proportionality, and then solving an ILP model to deal with rounding.

In order to define the quotas we may face the problem from two different points of view. On one hand we may consider the requirements 1–4 just as constraints and we try to minimize a measure of the deviation of the seats from the 'natural' quotas. On the other hand we may consider the constraints as directives which involve smoothly all countries and as a consequence the concept of quotas must be revised.

The note is organized as follows. In Section 2 the problem is defined in mathematical terms. The integer linear programming problem is presented in Section 3. The concept of quotas is revised in Section 4 and projective quotas are defined. Simpler affine quotas are defined in Section 5 together with divisor quotas derived by the method proposed in Grimmett et al. (2011), and the modified quotas proposed in Balinski and Young (2001). The results are briefly discussed in Section 6 and some conclusions follow in Section 7. Finally all results are displayed in tables at the end of the note.

2. Problem statement

There are n given countries. Their populations are $p_1, \dots, p_n$. We assume that the countries are sorted as $p_1 > p_2 > \cdots > p_n$ (we may freely assume that there are no countries with the same population). Let $P := \sum_i p_i$. The house size (total number of seats in the parliament) is H. The number of seats to be assigned to country i is x_i. The maximum number of seats for each country is M and the minimum number is m. The constraints to be satisfied by the seats are

$$x_1 \geq x_2 \geq \cdots \geq x_n \tag{1}$$

$$\frac{p_1}{x_1} > \frac{p_2}{x_2} > \cdots > \frac{p_n}{x_n} \tag{2}$$

$$x_1 \leq M, \qquad x_n \geq m \tag{3}$$

$$\sum_i x_i = H. \tag{4}$$

In the actual case of the EU Parliament the data are $m = 6$, $M = 96$, $H = 751$. The population data are reported in Table 1. The constraint (1) is the requirement 1. The constraints (2) is the degressive proportionality requirement 4. The requirements 2 and 3 are the bounds (3).

Note that constraints (1) and (2) together restrict considerably the range of possible solutions. If, for instance, we consider two variables x_1 and x_2, the feasible values are those included between the straight lines $x_2 = x_1$ and $x_2 = (p_2/p_1)\, x_1$. For p_2 close to p_1 this is a narrow cone which may not include integer points satisfying also (4). Indeed the counterexample provided in Grimmett et al. (2011) by extracting five European countries with almost equal populations shows that there can be no feasible solution at all.

The lower bound constraint in (3) is present in many actual laws (e.g., in the US House of Representatives we have $m = 1$). The upper bound in (3) may be questioned. However, this is a political decision which is in the same line as the degressive proportionality.

The seat allocation must satisfy the constraints (1)–(2)–(3)–(4) and at the same time be as proportional as possible to the respective populations. We address this problem by splitting these requirements into two separate problems.

On one hand the proportionality requirement leads to the definition of rational numbers, called *quotas*, which constitute the ideal proportional seat assignment if only seats were allowed to be fractional. There are many ways to define meaningful quotas. Note that, since the seats have to be close to the quotas, it makes sense to have the quotas themselves satisfying the constraints (1)–(2)–(3)–(4), although this is not strictly necessary. Perhaps the simplest way is to define the following *natural quotas*

$$q_i = p_i \frac{H}{P}.$$

However, the bounds (3) may be not satisfied by the natural quotas and the degressive proportionality constraints (2) are clearly never satisfied. Hence we need to possibly find other ways of defining the quotas for the EU Parliament.

The question we address for the quota definition is whether the seat allocation has simply to satisfy the constraints while trying to be as proportional as possible to the natural quotas, or the requirements imposed by the law, including the bounds, should have an effect which spreads over all countries, even those which should be unaffected by the constraints. This should be reflected in the quotas themselves. In Section 4 we pursue the second view.

Then the seat allocation within the stated constraints is taken care of by finding integer numbers satisfying the constraints and as close as possible to the defined quotas. This is explained in the next section.

3. An integer linear programming model

A first integer linear programming problem for the seat assignment minimizing the sum of deviations from given quotas q_i is the following:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} w_i \\
& w_i \geq q_i - x_i && i = 1, \dots, n \\
& w_i \geq x_i - q_i && i = 1, \dots, n \\
& x_i \geq x_{i+1} && i = 1, \dots, n-1 \\
& p_{i+1}\, x_i \leq p_i\, x_{i+1} && i = 1, \dots, n-1 \\
& \sum_{i=1}^{n} x_i = H \\
& x_1 \leq M, \quad x_n \geq m \\
& x_i \text{ integer} && i = 1, \dots, n.
\end{aligned} \tag{5}$$

A drawback of (5) is that optimal solutions are not guaranteed to be unique. For any two countries which are in defect (or in surplus) of seats with deviation larger than one for at least one country, we may exchange seats without altering the objective function value. For example, we might have $q_i = 16.3$ and $q_j = 15.6$ for two countries i and j. Assume also that $16\, p_i > 18\, p_j$. Then a feasible solution with $x_i = 17$ and $x_j = 17$ and another feasible solution with $x_i = 18$ and $x_j = 16$ (everything else the same) would have the same objective function value, since $w_i + w_j$ is the same in both cases. However, the first solution is clearly preferable because the maximum deviation is smaller.

In order to assign a larger penalty to larger deviations and still have a linear model we split each deviation w_i into smaller terms w_{ik} as $w_i = \sum_{k=1}^{K} w_{ik}$, with $0 \leq w_{ik} \leq 1$ (we have to figure out a number K of terms large enough to allow w_i to be split into the w_{ik}'s). Then each w_{ik} receives the penalty coefficient k. Due to these coefficients in the objective function, the deviation w_i will be first 'filled' by the terms with smallest k.