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# A dynamic programming algorithm for input estimation on linear time-variant systems

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**Abstract**

A time domain input estimation algorithm for linear systems with general time-varying parameters is developed. The algorithm is an extension of an existing approach for time-invariant state space models and several new features, such as higher order input approximations and an extended time-variant output relation including direct input influence, are introduced. Numerical examples are given to illustrate the new features and show that the algorithm is valid in a general time-variant setting. In particular, excellent results are obtained for an ill-posed moving force identification problem with noise-contaminated data, treated with Tikhonov regularization.


*Keywords:* Inverse problem; Indirect measurements; Time-variant systems; Time domain; Dynamic programming; State space

## 1. Introduction

Knowledge of time-varying excitation is of importance in the design of a wide range of engineering applications, from spacecraft and processing plants to electronic circuits. Regardless of the actual application or the underlying physics, the expected input will play a key role in the determination of adequate system properties or parameters. In a trivial case, the desired information may be obtained by direct measurement of the input. When this is not possible, the analyst may resort to indirect measurement techniques, which means that the unknown input is established as the solution to an inverse problem, based on measurements of system response. This process is known as input estimation.

In mechanical engineering, the concept of input estimation is mainly associated with determination of unknown dynamic forces acting on some kind of mechanical system. The nature of such forces will in many cases imply practical difficulties that prevent them from being measured directly. The loading positions may be inaccessible, the spatial distribution may be complicated, or the application of force transducers may intrude on the load path or alter system properties in an undesired manner. Various methods for solving the inverse problem associated with indirect force measurement have been proposed, see e.g. [1–3] for an overview. With few exceptions (e.g. [4,5]), these methods are concerned with linear systems. Several of the methods operate in the frequency domain or utilize modal system descriptions for computational efficiency, see e.g. [6,7], which is convenient for time-invariant systems. However, it is not very efficient to treat problems with time-varying system parameters with methods based on a modal approach, since the modal descriptions will change over time. Thus, it is often more convenient to treat such problems with time domain techniques. Recent work concerned with time-variant problems includes [8], where the conjugate gradient method is used to identify the excitation force on a

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single-degree-of-freedom system with time-varying stiffness and damping parameters, and [9] where dynamic programming is used to identify moving forces from strain and velocity measurements on a bridge model.

In this paper a non-iterative recurrence algorithm for input estimation on fully time-variant linear systems is presented. The algorithm is based on dynamic programming and is an extension of a time domain approach for time-invariant systems presented in [10]. The extended algorithm applies to problems where the explicit time-dependence of system and output coefficient matrices is known. For force estimation problems in mechanics, this implies that the force locations may vary but must be well-defined at all times. The method sets out from a system of ordinary differential equations in time, suggesting that any spatial variations in the physical problem have been parameterized, e.g. through finite element discretization. Thus, unknown excitations with spatial distribution should be discretized accordingly and consequently estimated in terms of the corresponding parameters. A general restriction, implied by the adopted temporal parametrization of the input, is that the number of sought inputs cannot exceed the number of sensors used.

## 2. Input estimation and dynamic programming

Input estimation based on dynamic programming, as described in [10], traditionally sets out from a linear, time-invariant system description on the discrete-time state space form

$$\boldsymbol{x}_{j+1} = \boldsymbol{A}\boldsymbol{x}_j + \boldsymbol{B}\boldsymbol{u}_j, \tag{1a}$$

$$\boldsymbol{y}_j = \boldsymbol{C}\boldsymbol{x}_j, \tag{1b}$$

where $\boldsymbol{A}$ is the plant matrix, $\boldsymbol{B}$ the input influence matrix and $\boldsymbol{C}$ the output influence matrix, all of which are independent of time and current state. The system has $n$ state variables $\boldsymbol{x}$, $n_{\mathrm{i}}$ inputs $\boldsymbol{u}$ and $n_{\mathrm{o}}$ outputs $\boldsymbol{y}$. The general idea is to seek the input history $\{\boldsymbol{u}_j^{\mathrm{opt}}\}_{j=1}^{N}$ that minimizes the weighted output residual

$$F = \sum_{j=1}^{N} (\boldsymbol{y}_j - \hat{\boldsymbol{y}}_j)^{\mathrm{T}} \boldsymbol{W}^{\mathrm{w}} (\boldsymbol{y}_j - \hat{\boldsymbol{y}}_j),$$

where $\hat{\boldsymbol{y}}_j$ denotes measurement data related to discrete-time instance $j$ and $\boldsymbol{W}^{\mathrm{w}}$ is a symmetric and positive definite weighting matrix. Since many problems are rank deficient or ill-conditioned (see [4]), a penalty term is usually added to the output residual, so that the objective function becomes

$$F = \sum_{j=1}^{N} (\boldsymbol{y}_j - \hat{\boldsymbol{y}}_j)^{\mathrm{T}} \boldsymbol{W}^{\mathrm{w}} (\boldsymbol{y}_j - \hat{\boldsymbol{y}}_j) + (\boldsymbol{u}_j)^{\mathrm{T}} \boldsymbol{W}^{\mathrm{r}} \boldsymbol{u}_j. \tag{2}$$

Addition of the penalty term is known as *Tikhonov regularization*, see [11], and allows restraints on the solution, such as smoothness, to be incorporated through a symmetric and positive (semi-)definite regularization matrix $\boldsymbol{W}^{\mathrm{r}}$. Regularization may for example be used to reduce the effects of measurement noise, cf. [12].

Dynamic programming is basically a recurrence algorithm that may be used to solve large least squares problems (see [13]) and, as such, it is clearly appropriate for minimizing quadratic functions like Eq. (2). The algorithm proposed in [10] has been shown to work well for input estimation problems where the system equations and output relations can be formulated according to Eqs. (1a)–(2), see for instance [3,10]. However, there are some important categories of linear input estimation problems to which the algorithm proposed in [10] cannot be applied, due to the following limitations:

(1) The system and output matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, as well as the weighting and regularization matrices, $\boldsymbol{W}^{\mathrm{r}}$ and $\boldsymbol{W}^{\mathrm{w}}$, are assumed to be time-invariant. This assumption discriminates some rather common input estimation problems from being solved with dynamic programming. Consider, for example, a case where the sampling interval varies. Even though the continuous-time system may be time-invariant, the corresponding discrete-time matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ will vary from one time interval to another. For moving load problems in mechanics the input influence matrix changes continuously, since the loading positions change. The use of dynamic programming for input estimation on either of these two cases requires a formulation capable of handling time-varying system descriptions. If, in addition, the output influence matrix $\boldsymbol{C}$ and the weighting matrix $\boldsymbol{W}^{\mathrm{w}}$ are allowed to vary with time, it is possible to use different measurements at different time instances $j$ (if so desired). Similarly, a potentially time-varying regularization matrix $\boldsymbol{W}^{\mathrm{r}}$ yields greater flexibility for imposing restraints on the solution.

(2) The output relation Eq. (1b) does not allow for direct coupling between inputs and outputs. So-called 'collocated' measurements, cf. [14], have the property that input has instant influence on output; consider for example an accelerometer whose location coincides with the position of a sought input force. Clearly, such measurements cannot be treated directly if an output relation according to Eq. (1b) is used. One way of solving this is to reformulate the system