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Controller synthesis for robust invariance of polynomial dynamical systems using linear programming☆

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ABSTRACT

In this paper, we consider a control synthesis problem for a class of polynomial dynamical systems subject to bounded disturbances and with input constraints. More precisely, we aim at synthesizing at the same time a controller and an invariant set for the controlled system under all admissible disturbances. We propose a computational method to solve this problem. Given a candidate polyhedral invariant, we show that controller synthesis can be formulated as an optimization problem involving polynomial cost functions over bounded polytopes for which effective linear programming relaxations can be obtained. Then, we propose an iterative approach to compute the controller and the polyhedral invariant jointly. Each iteration of the approach mainly consists in solving two linear programs (one for the controller and one for the invariant) and is thus computationally tractable. Finally, we show with several examples the usefulness of our method in applications.

1. Introduction

The design of nonlinear systems remains a challenging problem in control science. In the past decade, building on spectacular breakthroughs in optimization over polynomial functions [1,2], several computational methods have been developed for synthesizing controllers for polynomial dynamical systems [3,4]. These approaches have shown themselves to be successful for several synthesis problems such as stabilization or optimal control in which Lyapunov functions and cost functions can be represented or approximated by polynomials. However, these approaches are not suitable for some other problems, such as those involving polynomial dynamical systems with constraints on states and inputs, and subject to bounded disturbances.

In this paper, we consider a control synthesis problem for this class of systems. More precisely, given a polynomial dynamical system with input constraints and bounded disturbances, given a set of initial states $\underline{P}$ and a set of safe states $\overline{P}$, we aim at synthesizing a controller satisfying the input constraints and such that trajectories starting in $\underline{P}$ remain in $\overline{P}$ for all possible disturbances. This problem can be solved by computing jointly the controller and an invariant set for the controlled system which contains $\underline{P}$ and is included in $\overline{P}$ (see e.g. [5]). Here, we should mention that, even in the linear case, the problem of designing jointly a controller and an invariant is not trivial [6,7], and it is known that it can lead to a nonlinear problem.

In the following, we propose a computational method based on the use of parameterized template expressions for the controller and the invariant. Given a candidate polyhedral invariant, we show that controller synthesis can be formulated as an optimization problem involving polynomial objective functions over bounded polytopes. Recently, using various tools such as the blossoming principle [8] for polynomials, multi-affine functions [9], and Lagrangian duality, it has been shown how effective linear programming relaxations can be obtained for such optimization problems [10]; these relaxations were then used for the computation of invariants for autonomous polynomial dynamical systems. The improvement over the work of [10] is that, in this paper, constrained inputs and bounded disturbances are considered; also, an iterative approach is given to compute jointly a controller and a polyhedral invariant. Each iteration of the approach mainly consists in solving two linear programs and is thus computationally tractable. Finally, we show applications of our approach to several examples.

2. Problem formulation

In this work, we consider a nonlinear affine control system subject to input constraints and bounded disturbances:

$$\dot{x}(t) = f(x(t), d(t)) + g(x(t), d(t))u(t), \quad d(t) \in D,\ u(t) \in U, \tag{1}$$

where $x(t) \in R_X \subseteq \mathbb{R}^n$ is the state of the system, $d(t) \in D \subseteq \mathbb{R}^m$ is an external disturbance, and $u(t) \in U \subseteq \mathbb{R}^p$ is the control input.

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We assume that the vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and that the control matrix $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{(n \times p)}$, defining the dynamics of the system, are multivariate polynomial maps.

We also assume that the set of states is a bounded rectangular domain: $R_X = [\underline{x_1}, \overline{x_1}] \times \cdots \times [\underline{x_n}, \overline{x_n}]$, with $\underline{x_k} < \overline{x_k}$ for all $k \in \{1, \dots, n\}$; and that the set of disturbances D and the set of inputs U are convex compact polytopes:

$$D = \{d \in \mathbb{R}^m | \ \alpha_{D,k} \cdot d \leq \beta_{D,k}, \ \forall k \in \mathcal{K}_D\}$$
$$U = \{u \in \mathbb{R}^p | \ \alpha_{U,k} \cdot u \leq \beta_{U,k}, \ \forall k \in \mathcal{K}_U\},$$

where $\alpha_{D,k} \in \mathbb{R}^m$, $\beta_{D,k} \in \mathbb{R}$, $\alpha_{U,k} \in \mathbb{R}^p$, $\beta_{U,k} \in \mathbb{R}$, $\mathcal{K}_D$, and $\mathcal{K}_U$ are finite sets of indices.

We will denote by $R_D = [\underline{d_1}, \overline{d_1}] \times \cdots \times [\underline{d_m}, \overline{d_m}]$ the interval hull of polytope D, that is, the smallest rectangular domain containing D; and by $V_X = \{\underline{x_1}, \overline{x_1}\} \times \cdots \times \{\underline{x_n}, \overline{x_n}\}$ and $V_D = \{\underline{d_1}, \overline{d_1}\} \times \cdots \times \{\underline{d_m}, \overline{d_m}\}$ the sets of vertices of R_X and R_D. The present work deals with controller synthesis for a notion of invariance defined as follows.

Definition 1. Consider a set of states $P \subseteq R_X$ and a controller $h : R_X \to U$. The controlled system

$$\dot{x}(t) = f(x(t), d(t)) + g(x(t), d(t))h(x(t)), \quad d(t) \in D, \tag{2}$$

is said to be P-invariant if all trajectories with $x(0) \in P$ satisfy $x(t) \in P$ for all $t \geq 0$.

Let us remark that this is a notion of robust invariance, since it has to hold for all possible disturbances. Let $\underline{P} \subseteq \overline{P} \subseteq R_X$ be convex compact polytopes. In this paper, we consider the problem of synthesizing a controller h for system (1) such that all controlled trajectories starting in $\underline{P}$ remain in $\overline{P}$ forever. This can be seen as a safety property in which $\underline{P}$ is the set of initial states and $\overline{P}$ is the set of safe states. The problem can be solved by synthesizing jointly a controller and a polyhedral invariant $P \subseteq R_X$ containing $\underline{P}$ and included in $\overline{P}$.

Problem 1. Synthesize a controller $h : R_X \to U$ and a convex compact polytope P such that $\underline{P} \subseteq P \subseteq \overline{P}$ and the controlled system (2) is P-invariant.

In the following, we describe an approach to solve this problem. To restrict the search space, we shall use parameterized template expressions for the controller h and the invariant P. First, we will impose the orientation of the facets of polytope P by choosing normal vectors in the set $\{\gamma_k \in \mathbb{R}^n | \ k \in \mathcal{K}_X\}$, where $\mathcal{K}_X$ is a finite set of indices. Then, polytope P can be written under the form

$$P = \{x \in \mathbb{R}^n | \ \gamma_k \cdot x \leq \eta_k, \ \forall k \in \mathcal{K}_X\},$$

where the vector $\eta \in \mathbb{R}^{|\mathcal{K}_X|}$, to be determined, specifies the position of the facets. The facets of P are denoted by F_k for $k \in \mathcal{K}_X$, where $F_k = \{x \in \mathbb{R}^n | \ \gamma_k \cdot x = \eta_k \text{ and } \gamma_i \cdot x \leq \eta_i, \ \forall i \in \mathcal{K}_X \setminus \{k\}\}$. For simplicity, we will assume that the polytopes $\underline{P}$ and $\overline{P}$ are of the form

$$\underline{P} = \{x \in \mathbb{R}^n | \ \gamma_k \cdot x \leq \underline{\eta}_k, \ \forall k \in \mathcal{K}_X\}$$
$$\overline{P} = \{x \in \mathbb{R}^n | \ \gamma_k \cdot x \leq \overline{\eta}_k, \ \forall k \in \mathcal{K}_X\}.$$

Then, the condition $\underline{P} \subseteq P \subseteq \overline{P}$ translates to $\underline{\eta}_k \leq \eta_k \leq \overline{\eta}_k$, $\forall k \in \mathcal{K}_X$.

Second, we will search for the controller h in a subspace spanned by a polynomial matrix:

$$h(x) = H(x)\theta,$$

where $\theta \in \mathbb{R}^q$ is a parameter to be determined and the matrix $H : \mathbb{R}^n \to \mathbb{R}^{(p \times q)}$ is a given multivariate polynomial map. The use of a template expression is natural when searching for a controller with a particular structure. The input constraint (i.e. for all $x \in R_X$, $h(x) \in U$) is then equivalent to

$$\forall k \in \mathcal{K}_U, \ \forall x \in R_X, \quad \alpha_{U,k} \cdot H(x)\theta \leq \beta_{U,k}. \tag{3}$$

Under these assumptions, the dynamics of the controlled system (2) can be rewritten under the form

$$\dot{x}(t) = f(x(t), d(t)) + G(x(t), d(t))\theta, \quad d(t) \in D,$$

where the matrix of polynomials $G(x, d) = g(x, d)H(x)$. From the standard characterization of invariant sets (see [11]), it follows that the controlled system (2) is P-invariant if and only if

$$\forall k \in \mathcal{K}_X, \ \forall x \in F_k, \ \forall d \in D,$$
$$\gamma_k \cdot (f(x, d) + G(x, d)\theta) \leq 0. \tag{4}$$

Then, Problem 1 can be solved by computing vectors $\theta \in \mathbb{R}^q$ and $\eta \in \mathbb{R}^{|\mathcal{K}_X|}$ with $\underline{\eta}_k \leq \eta_k \leq \overline{\eta}_k$ for all $k \in \mathcal{K}_X$, and such that (3) and (4) hold. In the following, we first show how, given the vector $\eta \in \mathbb{R}^{|\mathcal{K}_X|}$ (and hence the polytope P), we can compute, using linear programming, the parameter θ (and hence the controller h) such that the controlled system (2) is P-invariant. Then, we show how to compute jointly the controller h and the polytope P using an iterative approach based on sensitivity analysis of linear programs. Before that, we shall review some recent results on linear relaxations for optimization of polynomials over bounded polytopes [10].

3. Optimization of polynomials over polytopes

In this section, we review some recent results of [10] that will be useful for solving Problem 1. Let us consider the following optimization problem involving a polynomial on a bounded polytope:

$$\begin{array}{ll} \text{minimize} & c \cdot p(y) \\ \text{over} & y \in R, \\ \text{subject to} & a_i \cdot y \leq b_i, \quad i \in I, \\ & a_j \cdot y = b_j, \quad j \in J, \end{array} \tag{5}$$

where $p : \mathbb{R}^m \to \mathbb{R}^n$ is a multivariate polynomial map, $c \in \mathbb{R}^n$, $R = [\underline{y_1}, \overline{y_1}] \times \cdots \times [\underline{y_m}, \overline{y_m}]$ is a rectangle of $\mathbb{R}^m$; I and J are finite sets of indices; $a_k \in \mathbb{R}^m$, $b_k \in \mathbb{R}$, for all $k \in I \cup J$. Let us remark that, even though the polytope defined by the constraints indexed by I and J is unbounded in $\mathbb{R}^m$, the fact that we consider $y \in R$ which is a bounded rectangle of $\mathbb{R}^m$ results in an optimization problem on a bounded (not necessarily full dimensional) polytope of $\mathbb{R}^m$. Let p^* denote the optimal value of problem (5). The approach presented in [10] allows us to compute a guaranteed lower bound d^* of p^*. The approach is as follows. First, using the so-called blossoming principle [8], we transform problem (5) into an equivalent optimization problem involving a multi-affine function on a polytope. The dual of this problem is then a linear program that is easily solvable and whose optimal value is a guaranteed lower bound of p^*.

3.1. Blossoming principle

Multi-affine functions form a particular class of multivariate polynomials. Essentially, a multi-affine function is a function which is affine in each of its variables when the other variables are regarded as constant.

Definition 2. A multi-affine function $q : \mathbb{R}^M \to \mathbb{R}$ is a multivariate polynomial in the variables $z_1, \dots, z_M$ where the degree of g in each of its variables is at most 1:

$$q(z) = q(z_1, \dots, z_M) = \sum_{(d_1,\dots,d_M) \in \{0,1\}^M} q_{(d_1,\dots,d_M)} z_1^{d_1} \dots z_M^{d_M},$$