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A dynamic programming approach for pricing options embedded in bonds

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Abstract

We propose a dynamic programming (DP) approach for pricing options embedded in bonds, the focus being on call and put options with advance notice. An efficient procedure is developed for the cases where the interest-rate process follows the Vasicek, Cox–Ingersoll–Ross (CIR), or generalized Vasicek models. Our DP methodology uses the exact joint distribution of the interest rate and integrated interest rate at a future date, conditional on the current value of the interest rate. We provide numerical illustrations, for the Vasicek and CIR models, comparing our DP method with finite-difference methods. Our procedure compares quite favorably in terms of both efficiency and accuracy. An important advantage of the our DP approach is that it can be applied to more general models calibrated to capture the term structure of interest rates (e.g., the generalized Vasicek model).

JEL classification: C63; G13

Keywords: Option pricing; Bond pricing; Dynamic programming; Options embedded in bonds

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0. Introduction

A *bond* is a contract that pays its holder a known amount, the *principal*, at a known future date, called the *maturity*. The bond may also pay periodically to its holder fixed cash dividends, called the *coupons*. When it gives no dividend, it is known as a *zero-coupon bond*. A bond can be interpreted as a loan with a known principal and interest payments equal to the coupons (if any). The borrower is the *issuer* of the bond and the lender, i.e., the holder of the bond, is the *investor*.

Several bonds contain one or several options coming in various flavors. The *call option* gives the issuer the right to purchase back its debt for a known amount, the *call price*, during a specified period within the bond's life. Several government bonds contain a call feature (see Bliss and Ronn, 1995 for the history of callable US Treasury bonds from 1917). The *put option* gives the investor the right to return the bond to the issuer for a known amount, the *put price*, during a specified period within the bond's life. These options are an integral part of a bond, and cannot be traded alone, as is the case for call and put options on stocks (for example). They are said to be *embedded* in the bond. In general, they are of the *American-type* and, thus, allow for early exercising, so that the bond with its embedded options can be interpreted as an American-style interest-rate derivative. This paper focuses on call and put options embedded in bonds with *advance notice*, that is, options with exercise decisions prior to exercise benefits.

As it is often the case in practice, we assume that exercise of the call and put options is limited to the coupon dates posterior to a known *protection period* and that there is a *notice period* of fixed duration Δt. Thus, consider a coupon date t_m where the exercise is possible, and let C_m and P_m be the *call* and *put prices* at t_m. The decision to exercise or not by the issuer and the investor must be taken at $t_m - \Delta t$. If the issuer calls back the bond at $t_m - \Delta t$, he pays C_m to the investor at t_m, and, similarly, if the investor puts the bond at $t_m - \Delta t$, he receives P_m from the issuer at t_m. If no option is exercised at t_m, by the no-arbitrage principle of asset-pricing (Elliott and Kopp, 1999), the value of the bond is equal to the expected value of the bond at the next decision date, discounted at the interest rate. This expectation is taken under the so-called *risk-neutral probability measure*, where the uncertainty lies in the future trajectory of the (risk-free short-term) interest rate.

There are no analytical formulas for valuing American derivatives, even under very simplified assumptions. Numerical methods, essentially trees and finite differences, are usually used for pricing. Recall that trees are numerical representations of discrete-time and finite-space models and finite differences are numerical solution methods for partial differential equations. The pricing of American financial derivatives can also be formulated as a Markov decision process, that is, a stochastic *dynamic programming* (DP) problem, as pointed out by Barraquand and Martineau (1995). Here the DP value function, that is, the value of the bond with its embedded options, is a function of the current time and of the current interest rate, namely the *state variable*. This value function verifies a DP recurrence via the no-arbitrage principle of asset pricing, the solution of which yields both the bond value and the