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Dynamic programming for impulse controls ☆

A.B. Kurzhanski *, A.N. Daryin

Moscow State (Lomonosov) University, Faculty of Computational Mathematics and Cybernetics, Russia

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Abstract

This paper describes the theory of feedback control in the class of inputs which allow delta-functions and their derivatives. It indicates a modification of dynamic programming techniques appropriate for such problems. Introduced are physically realizable bang-bang-type approximations of the "ideal" impulse-type solutions. These may also serve as "fast" feedback controls which solve the terminal control problem in arbitrary small time. Examples of damping high-order oscillations in finite time are presented.

Keywords: Closed-loop impulse control; Feedback strategy; Dynamic programming; Variational inequality; Generalized controls; High-order impulses; Nonlinear analysis; Bang-bang control; Fast controls; Oscillating systems; Finite time

1. Introduction

As well known, problems of impulse control had been among the topics of control theory since its creation, where they were mostly treated as those of open-loop control (Krasovski, 1957; Neustadt, 1964). However, many recent applied motivations (hybrid systems, coordinated control, communication for control, etc.), also require and justify the application of impulsive inputs. But now the request is to deal with closed-loop schemes. Similar mathematical problems, arising in economic models, were indicated by Bensoussan and Lions (1982).

In contrast with most previous investigations, this paper deals with the problem of *closed-loop impulse control* based on generalization of dynamic programming techniques in the form of variational inequalities of the Hamilton–Jacobi–Bellman (HJB) type. Once subjected to closed-loop impulse controls, the originally linear systems treated here, become *nonlinear.* A special feature, described in this paper, is the application of higher order impulses which are derivatives of delta-functions, introduced for open-loop controls by Kurzhanski and Osipov (1969). Such "ideal" controls allow to transfer a controllable system from one state to another in zero time. They may also serve as virtual controls for system resets in hybrid system models. However, these ideal impulse controls are not physically realizable. In order to ensure their applicability, a scheme for substituting them by realizable approximations is introduced, which leads to the description of "fast" controls that can solve problems of terminal control in arbitrary small time. This paper gives a concise description of related theory with examples on damping high-order oscillating systems to zero in finite time. It is based on two presentations at IFAC Conferences (Daryin & Kurzhanskii, 2007b; Kurzhanski, 2007), see also Daryin, Kurzhanski, and Seleznev (2005), Kurzhanski (2006), Daryin and Kurzhanskii (2007a).

2. The impulse control problem

Consider the following generalization of the Mayer–Bolza problem.

Problem 1. Minimize the functional

$$J(u(\cdot)) = \operatorname{Var} U(\cdot) + \varphi(x(\vartheta + 0)) \to \inf, \tag{1}$$

$$dx(t) = A(t)x(t)\,dt + B(t)\,dU(t), \quad t \in [t_0, \vartheta], \tag{2}$$

under restriction

$$x(t_0) = x^0. \tag{3}$$

Here Var $U(\cdot)$ is the *total variation* of function $U(\cdot)$, over the interval $[t_0,\vartheta]$, where $U(\cdot) \in \mathbf{V}^p[t_0,\vartheta]$ is the space of vector-valued functions of bounded variation. The *generalized control* $U(t)$ attains its values in $\mathbb{R}^p$ (which means that each component $U_i(t)$ is a function of bounded variation on $[t_0,\vartheta]$). The matrix functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times p}$ are continuous (or k times

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* Corresponding author. Tel.: +7 495 932 8850; fax: +7 495 932 8850.
E-mail address: kurzhans@mail.ru (A.B. Kurzhanski).

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differentiable, $k \leq n-1$, if the solution allows not only delta-functions but also their derivatives of order up to k).

Eq. (2) with condition (3) is a symbolic relation for[1]

$$x(t) = G(t, t_0)x^0 + \int_{t_0}^{t} G(t, \xi)B(\xi)\, dU(\xi),$$

where the last term in the right-hand side is a Stieltjes integral. The terminal time is fixed and the terminal cost function $\varphi(x) : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is closed and convex. We assume that functions $x(t)$ and $U(t)$ are *left-continuous*.

A special selection of $\varphi(x) = \mathcal{I}(x|\{x^1\})$ yields the problem of steering $x(t)$ from point $x^0 = x(t_0)$ to point $x^1 = x(\vartheta)$ with minimal variation of the control $U(t)$:

$$\operatorname{Var} U(\cdot) \to \inf,$$

due to system (3), under boundary conditions

$$x(t_0) = x^0, \quad x(\vartheta + 0) = x^1. \tag{4}$$

Problems of the described type have been solved in detail as those of *open-loop control* in the class of controls that allow delta-functions of first order (see Krasovski, 1968; Neustadt, 1964) and those that allow higher derivatives of delta-functions (see Gusev, 1975; Kurzhanski, 1975; Kurzhanski & Osipov, 1969). In the present section we will however seek for solutions in the class of closed-loop feedback controls. Such a turn would require to develop a dynamic programming approach (see also Miller and Rubinovich (2003), Motta and Rampazzo (1996)). But prior to that we will first indicate the open-loop solutions in the class which allows delta-functions but not their derivatives.

3. The open-loop control

Define

$$V(t_0, x^0) = \min\{J(u(\cdot))|x(t_0) = x^0\}$$

as the *value function* for Problem 1. We shall also use notation $V(t_0, x^0) = V(t_0, x^0; \vartheta, \varphi(\cdot))$, emphasizing the dependence on $\{\vartheta, \varphi(\cdot)\}$.

Let us start by minimizing

$$V_1(t_0, x^0) = V(t_0, x^0; \vartheta, \mathcal{I}(\cdot|\{x^1\})).$$

Then we first find conditions for solvability of the boundary-value Problem 4 under constraint $\operatorname{Var} U(\cdot) \leq \mu$, with μ given.

Since

$$\langle \ell, x(\vartheta + 0)\rangle = \langle \ell, x^1\rangle \leq \langle \ell, G(\vartheta, t_0)x^0\rangle + \max\left\{ \int_{t_0}^{\vartheta} \langle \ell, G(\vartheta, \xi)B(\xi)\, dU(\xi)\rangle | \operatorname{Var} U(\cdot) \leq \mu \right\},$$

and since the conjugate space for $\mathcal{C}^p[t_0, \vartheta]$ is $\mathbf{V}^p[t_0, \vartheta]$, then, treating functions $\langle \ell, G(\vartheta, t)B(t)\rangle$ as elements of $\mathcal{C}^p[t_0, \vartheta]$, we have

$$\langle \ell, x^1\rangle \leq \langle \ell, G(\vartheta, t_0)x^0\rangle + \mu\|B^T(\cdot)G^T(\vartheta, \cdot)\ell\|_{\mathcal{C}[t_0,\vartheta]}, \tag{5}$$

whatever be $\ell \in \mathbb{R}^n$.

Here

$$\|B^T(\cdot)G^T(\vartheta, \cdot)\ell\|_{\mathcal{C}[t_0,\vartheta]} = \max_{t \in [t_0,\vartheta]} \|B^T(t)G^T(\vartheta, t)\ell\| = \|\ell\|_V,$$

where $\|\ell\|$ is the Euclidean norm.

Theorem 1. *The problem of steering $x(t)$ from point $x^0 = x(t_0)$ to point $x^1 = x(\vartheta)$ is solvable iff* $\operatorname{Var}\{U(\cdot)|[t_0, \vartheta]\} \leq \mu$.

The optimal control $U^0(t)$ for this problem is of minimal variation

$$\mu^0 = \sup\left\{ \frac{\langle \ell, x^1 - G(\vartheta, t_0)x^0\rangle}{\|B^T(\cdot)G^T(\vartheta, \cdot)\ell\|_{\mathcal{C}[t_0,\vartheta]}} | \ell \in \mathbb{R}^n \right\}. \tag{6}$$

Lemma 1. *The maximum in (6) is attainable under the following assumption: system (2) is completely controllable.*

Indeed, if $\mu^0 \neq 0$ and the controllability assumption is true, then $\|\ell\|_V$ defines a finite-dimensional norm in $\mathbb{R}^n$ and (6) is equivalent to

$$\mu^0 = \max_{\|\ell\|_V \leq 1} \langle \ell, x^1 - G(\vartheta, t_0)x^0\rangle = \|x^1 - G(\vartheta, t_0)x^0\|_V^*, \tag{7}$$

where $\|x\|_V^*$ is the finite-dimensional norm conjugate to $\|\ell\|_V$.

Remark 1. Note that under the controllability assumption we have $\mu^0 = \mu^0(t_0, x^0; \vartheta, x^1) < \infty$.

Let ℓ^0 be the maximizer in (7). Then from (5) and (6) we observe

$$\mu^0 = \max\left\{ \int_{t_0}^{\vartheta} \langle \ell^0, G(\vartheta, \xi)B(\xi)\, dU(\xi)\rangle | \operatorname{Var} U(\cdot) \leq \mu^0 \right\} = \int_{t_0}^{\vartheta} \langle \ell^0, G(\vartheta, \xi)B(\xi)\, dU^0(\xi)\rangle. \tag{8}$$

This is *the maximum principle for impulse controls which is given in the integral form.*

Using the notation $\psi[t] = G^T(\vartheta,t)\ell = \psi(\xi, \vartheta, \ell)$ with $\psi^0[\xi] = \psi(\xi, \vartheta, \ell^0)$, we can rewrite (8) as

$$\mu^0 = \max\left\{ \int_{t_0}^{\vartheta} \langle \psi^0[\xi], B(\xi)\, dU(\xi)\rangle | \operatorname{Var} U(\cdot) \leq \mu^0 \right\} = \int_{t_0}^{\vartheta} \langle \psi^0[\xi], B(\xi)\, dU^0(\xi)\rangle. \tag{9}$$

Theorem 2. *The optimal impulse control $U_0(\cdot)$ that minimizes the variation* $\operatorname{Var} U(\cdot)$ *satisfies the maximum principle (9). With $\mu^0 > 0$ and under the controllability assumption* (9) *is also sufficient for the optimality of $U^0(\cdot)$.*

Remark 2. The optimal control $U^0(\cdot)$ of Problem 1 may not be unique.

Denote $\mathcal{F}(\ell) = \operatorname{argmax}_t\{\|B^T(t)G^T(\vartheta, t)\ell\| \; | t \in [t_0, \vartheta]\}$, which is the set of points τ where

$$\|B^T(\tau)G^T(\vartheta, \tau)\ell\| = \max_t\{\|B^T(t)G^T(\vartheta, t)\ell\| \; | t \in [t_0, \vartheta]\}.$$

[1] Here $G(t, \tau)$ is the fundamental matrix of the system without control, i.e. the solution to the matrix ODE $\partial G(t, \tau)/\partial t = A(t)G(t, \tau)$ with initial condition $G(\tau, \tau) = I$.