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A study on the relaxed linear programming bounds method for system reliability

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A B S T R A C T

The linear programming (LP) bounds method was applied for computing bounds on the system reliability of general systems based on the individual component state probabilities and joint probabilities of the states of a small number of components. In the LP bounds method, the bounds of the system reliability can be obtained by using LP. These bounds are useful approximations when exact solutions are costly or unavailable. However, the size of the LP problem determined by the number of design variables and the number of constraints increases exponentially with the number of components. This size problem is the main drawback of the LP bounds method. This paper presents a relaxed linear programming (RLP) bounds method to overcome this drawback of the LP bounds method. The accuracy and efficiency of the RLP bounds method are investigated using numerical examples involving series and parallel systems.

1. Introduction

A system, in general, consists of a number of components, and the state of the system depends on the states of its constituent components. The probability that a system is in a particular functioning state (system reliability), or its complement (system failure probability), can then be expressed based on the probability of the component states. Computation of this probability, however, is extremely difficult, particularly when there exists a dependency among the component states and when the number of components is large.

The idea of using linear programming (LP) to compute bounds on system reliability was first explored by Hailperin [1]. Kounias and Marin [2] used the approach to examine the accuracy of some theoretical bounds. Later, specialized versions of this approach were employed in fields such as operations research [3]. Song and Der Kiureghian proposed the linear programming (LP) bounds method for computing the bounds on the failure probability of general systems based on the joint probabilities of the states of k components (when $k = 1$, these joint probabilities become the individual component state probabilities) [4]. The LP formulation has a number of important advantages over other existing methods (e.g., Boole bounds [5] or Zhang bounds [6]). They include: (a) any "level," i.e., the number (k) of components considered in the joint probabilities of the states, of information can be used, including equalities and inequalities; (b) the statistical dependency among component states is easily accounted for in terms of their joint probabilities; (c) the method guarantees the narrowest possible bounds for the given information of individual and joint component states probabilities; (d) the method is applicable to a general system, including a system that is neither pure series nor pure parallel and a system for which no theoretical formula exists; and (e) critical components and cut sets within a system can be easily identified [7].

There exists, however, a critical drawback in the LP bounds method. The size of the LP problem, which is usually determined by the number of design variables and the number of constraints, increases exponentially with the number of components. For a system with n two-state components, the number of design variables in the LP bounds method is $N_d = 2^n$. When $n = 17$, $N_d = 131\,072$ and the problem can be solved with ordinary LP solvers on a PC. When $n = 100$, this number becomes $N_d \approx 1.27 \times 10^{30}$, which is enormously large. The number of constraints, which depends on the number of design variables and the level of joint state probabilities, also becomes enormously large in the application of the LP bounds method to a large system. This size issue—both the number of design variables and constraints—would be a hindrance in the application of the LP bounds method to a large system.

To overcome the size issue of design variables, Der Kiureghian and Song propose a multi-scale approach, whereby the system is decomposed into subsystems and a hierarchy of analysis is performed by considering each subsystem or set of subsystems separately [7]. The decomposition facilitates solution of the system reliability by the LP bounds method, whereby the large LP problems for the entire system is replaced by several LP problems of much smaller size. This facility, however, comes at a cost; the system bounds computed for the decomposed system can be wider

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than the bounds computed for the intact system with the same level of probability information.

This paper proposes a relaxed linear programming (RLP) bounds method to overcome the size problem of the LP bounds method. The RLP bounds method employs the universal generating function to reduce the number of design variables from 2^n to $n^2 - n + 2$. The number of constraints can also be reduced substantially. The accuracy and efficiency of the RLP bounds method are investigated using numerical examples involving series and parallel systems.

2. Review of the linear programming bounds method

Considering a system with n two-state components, Hailperin [1] divided the sample space of component states into 2^n mutually exclusive and collectively exhaustive (MECE) events, each consisting of a distinct intersection of the failure events F_i and their complements $\bar{F}_i$ (functional events), i = 1, 2,…, n. We call them the basic MECE events and denote them by e_r, $r = 1, 2, \ldots, 2^n$. For example, when n = 3, one finds 2^3 = 8 basic MECE events to be $e_1 = F_1 \cap F_2 \cap F_3$, $e_2 = \bar{F}_1 \cap F_2 \cap F_3$, $e_3 = F_1 \cap \bar{F}_2 \cap F_3$, $e_4 = F_1 \cap F_2 \cap \bar{F}_3$, $e_5 = \bar{F}_1 \cap \bar{F}_2 \cap F_3$, $e_6 = \bar{F}_1 \cap F_2 \cap \bar{F}_3$, $e_7 = F_1 \cap \bar{F}_2 \cap \bar{F}_3$, and $e_8 = \bar{F}_1 \cap \bar{F}_2 \cap \bar{F}_3$ (see Fig. 1). It should be noted that only the joint failure probability of k components such as $P(F_i)$ and $P(F_i \cap F_j)$, $i, j = 1, 2, \ldots, n$, and $i \neq j$, is known, but any probability of the basic MECE event e_r, $p_{m_r} = P(e_r)$, is not known in advance.

Because of the mutual exclusivity of the basic MECE events, the probability of any union of these events is the sum of the corresponding probabilities. In particular, the probability of any failure event F_i is the sum of the probabilities of the basic MECE events that constitute the event F_i. For example, for the system with three components mentioned above, the component failure probability is expressed as

$$\begin{aligned} P(F_1) &= P_1 = p_{m_1} + p_{m_3} + p_{m_4} + p_{m_7} \\ P(F_2) &= P_2 = p_{m_1} + p_{m_2} + p_{m_4} + p_{m_6} \\ P(F_3) &= P_3 = p_{m_1} + p_{m_2} + p_{m_3} + p_{m_5} \end{aligned} \tag{1}$$

Similarly, any joint failure probability is given as the sum of the basic MECE events that constitute the intersection events. More generally, we write

$$\begin{aligned} P(F_i) &= P_i = \sum_{m_r: e_r \subseteq F_i} p_{m_r} \\ P(F_i \cap F_j) &= P_{ij} = \sum_{m_r: e_r \subseteq F_i \cap F_j} p_{m_r} \\ P(F_i \cap F_j \cap F_l) &= P_{ijl} = \sum_{m_r: e_r \subseteq F_i \cap F_j \cap F_l} p_{m_r}, \ \ etc. \end{aligned} \tag{2}$$

According to the basic axioms of probability, the above probabilities $p_m = \{p_{m_1}, p_{m_2}, \ldots, p_{m_{2^n}}\}$ are subject to the following linear constraints:

$$\sum_{r=1}^{2^n} p_{m_r} = 1 \tag{3}$$

$$p_{m_r} \geqslant 0; \ \ r = 1.2, \ldots, 2^n \tag{4}$$

The lower bound and the upper bound of the system failure probability is obtained as the minimum and the maximum of the objective function of the LP, respectively. The formulation of LP appropriate for this analysis has the following form:

$$\begin{aligned} &\text{minimize(maximize)} \quad \boldsymbol{c}^T \boldsymbol{p_m} \\ &\text{subject to} \quad \boldsymbol{a_1 p_m} = \boldsymbol{b_1} \\ &\qquad\qquad\quad \boldsymbol{a_2 p_m} \geq \boldsymbol{b_2} \end{aligned} \tag{5}$$

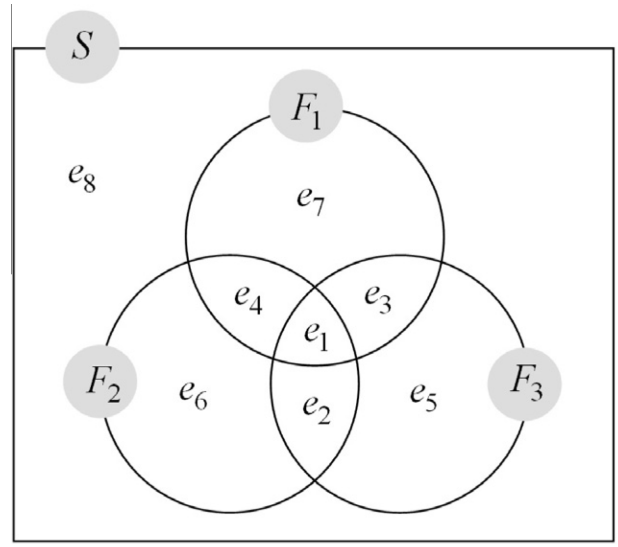

Fig. 1. The basic MECE event e_r for a three-event sample space.

where $\boldsymbol{p_m} = \{p_{m_1}, p_{m_2}, \ldots, p_{m_{2^n}}\}$ is the vector of design variables and represents the probabilities of the basic MECE events; $\boldsymbol{c}$ relates the system failure event with the component failure events; $\boldsymbol{c}^T\boldsymbol{p_m}$ is the linear objective function. $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ are the coefficient matrices; $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ are the coefficient vectors. These matrices and vectors represent the information given in terms of joint failure probabilities of k components. $\boldsymbol{a_1}$ and $\boldsymbol{b_1}$ are obtained from Eq. (2). $\boldsymbol{a_2}$ and $\boldsymbol{b_2}$ are also obtained from Eq. (2) when one has information such as $P(F_i) \geq x$ rather than $P(F_i) = x$. Additional linear constraints are imposed by the axioms of probability (Eqs. (3) and (4)) [4].

For the above three component system, if one knows $P(F_1)$ = 0.01, $P(F_2)$ = 0.02, and $P(F_3)$ = 0.03, and the objective function is $P(F_1 \cap F_2 \cap F_3) = p_{m_1}$, then $\boldsymbol{a_1}$ and $\boldsymbol{b_1}$ based on Eq. (2) and $\boldsymbol{c}^T$ are expressed as

$$\boldsymbol{a_1} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

$$\boldsymbol{b_1} = \begin{bmatrix} 0.01 \\ 0.02 \\ 0.03 \end{bmatrix} \tag{7}$$

$$\boldsymbol{c}^T = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0] \tag{8}$$

For a system with n components, the number of design variables (N_d) is 2^n. There are also one equality and 2^n inequality constraints resulting from the probability axioms Eqs. (3) and (4), respectively, and there are $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$ equality or inequality constraints resulting from Eq. (2) when the complete set of joint failure probabilities of each k components, i.e., the joint failure probabilities of all combinations up to each k components, is available. Thus, the total number of constraints of the LP bounds method, N_c, can be expressed as

$$N_c = 2^n + 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} \tag{9}$$

Note that it is not necessary to have the complete set of joint failure probabilities of k components at a particular level. Any partial set of joint failure probabilities of k components can be used. The bounds of failure probability of a system with an incomplete set of joint failure probabilities of k components information can be found in Example 1 (see Section 4).