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Shape sensitivity analysis in linear elastic fracture mechanics

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Abstract

Shape sensitivity analysis of an elastic solid in equilibrium with a known load system applied over its boundary is presented in this work. The domain and boundary integral expressions of the first- and second-order shape derivatives of the total potential energy are established, by using an arbitrary change of the domain characterized by a velocity field defined over the initial body configuration. In these expressions we recognize free divergence tensors that are denoted in this paper as energy shape change tensors. Next, shape sensitivity analysis is applied to cracked bodies. For that purpose, a suitable velocity distribution field is adopted to simulate the crack advance of a unit length in a two-dimensional body. Finally, the corresponding domain and the equivalent path-independent integral expressions of the first- and second-order potential energy release rate of fracture mechanics are also derived.

Keywords: Variational formulation; Shape sensitivity analysis; Fracture mechanics

1. Introduction

The interest in structural optimization and sensitivity analysis has increased greatly during the last decades due to the advent of reliable general numerical methods and computer power. More recently, researchers have focused their attention on one important and perhaps more complex class of engineering design problem, in which the shape, or more specifically, the domain over which the problem is defined is to be determined [1–3].

For such problem, known in the literature as shape sensitivity analysis and shape optimization, the domain becomes the design variable. Contributions to this field have been made using two fundamentally different approaches.

The first approach uses a discretized numerical model based either on the finite element method, the boundary element method, or other available numerical method, to carry out a shape design sensitivity analysis controlling either finite element or boundary element node motion and reducing the shape derivatives into differentiations of algebraic equations [4,5].

The second approach, which is adopted in the present work, resorts to a variational formulation of continuum mechanics [6,7]. In this case, well-known as continuum formulation, the shape design sensitivity analysis is carried out by controlling a velocity field which is introduced to simulate the shape change of the initial domain [8,9]. The variational equilibrium equation associated with the kinematical model is taken as the state equation and the total potential energy stored in an elastic body is chosen as the cost function, using the optimization nomenclature.

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The immediate difficulty in the application of this latter procedure is that the domain of integration is variable and both the state equation and cost function do not exhibit an explicit dependence on the project variable. To obviate this mathematical difficulty, the domain variation is parameterized and an analogy is established between the shape change and the motion of the deforming solid using the concept of material derivative of continuum mechanics to account for these changes in shape [10].

Although the second approach presented and adopted in this work poses major complexity regarding both physical and mathematical aspects, it nevertheless possesses the advantage of yielding the exact general expressions of the derivatives, which are independent of the approximate method used for the analysis.

Through this approach, integral expressions are derived for first- and second-order shape sensitivity analysis in terms of changes in the shape of the domain [11–16]. Furthermore, the condition of null divergence of the energy shape change tensors is employed to transform the domain integrals to boundary integrals, so that shape design sensitivity expressions can be obtained in terms of a shape perturbation of the boundary.

Once the derivatives of the total potential energy stored in an elastic body have been obtained, this analysis can be easily extended to the study of fracture mechanics [17,18]. For simplicity, we consider a two-dimensional body with a straight crack, under plane-strain state condition, subjected to a loading system on its boundary and simulate the initiation of crack advance as a suitable shape change of the body.

By selecting a distribution pattern of adequate velocities in the initial domain of the cracked body, we can specialize the expressions of the first and second derivatives of the potential energy to obtain the total potential energy release rates and the corresponding first- and second-order path-independent integrals.

2. Shape change

As a first step in the study of the behavior of function and functionals when the shape of a body is modified, we shall introduce in the present section the concept of shape variation or shape change of a body.

Let us identify the initial shape of the body with the three-dimensional domain $\Omega \subset R^3$ bounded by $\partial\Omega$. We assume that the body is under a given traction over part of the boundary denoted as $\partial\Omega_t$ and a prescribed displacement, null for simplicity, over the remaining part of the boundary $\partial\Omega_u$ ($\partial\Omega = \partial\Omega_t \cup \partial\Omega_u$).

We characterize the shape change of the body as the change of the initial domain Ω, described by the scalar parameter τ and the known, sufficiently smooth vector field $\mathbf{v} = \mathbf{v}(\mathbf{x})$, that defines the transformation from Ω to Ω_τ

$$\mathbf{x}_\tau = \mathbf{x}_\tau(\mathbf{x}) = \mathbf{x} + \tau\mathbf{v}(\mathbf{x}) \tag{1}$$

in which the subscript τ denotes the dependence of this parameter.

The transformed domain Ω_τ might be considered as a perturbation of the initial domain Ω and the transformation from Ω to Ω_τ as a function of the point $\mathbf{x}$ and the parameter τ

$$\Omega \mapsto \Omega_\tau, \quad \partial\Omega \mapsto \partial\Omega_\tau, \quad \mathbf{x} \mapsto \mathbf{x}_\tau. \tag{2}$$

Thus, shape change is simulated by a smooth one-parameter family of transformations, with τ being the parameter. Furthermore, introducing continuum mechanics terminology, an analogy can be drawn between change of shape and motion of a body.

The domain Ω_τ for different values of the parameter τ, which characterizes the shape change, is known in continuum mechanics as the trajectory of the body motion [19,20]. In this sense Ω_τ can be seen as the place occupied by Ω at time τ. When $\tau = 0$, the domain Ω_τ reduces to the initial domain Ω_0, which we denote Ω for simplicity. Thus, we may write

$$\Omega_\tau = \{\mathbf{x}_\tau = \mathbf{x} + \tau\mathbf{v}(\mathbf{x}),\ \mathbf{x} \in \Omega, \tau \in R^+\}. \tag{3}$$

Since at each τ, the shape change is a one-to-one transformation of Ω to Ω_τ, there is a unique inverse transformation from Ω_τ to Ω