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On a reliable solution of a quasilinear elliptic equation with uncertain coefficients: sensitivity analysis and numerical examples

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1. Introduction

The aim of the paper is to add sensitivity analysis and numerical tests to the existence and convergence results published in [2]. The isotropic material case is studied in [1]. By way of contrast, anisotropic medium is considered in this paper.

The mathematical problem examined in the paper has a clear physical meaning. In the words of physics, we can say we consider a steady-state heat flow in an anisotropic body. The temperature distribution is modeled by a quasilinear elliptic equation with uncertain coefficients of heat conductivity. These are temperature dependent and belong to an admissible set derived from measurements, for example. We choose a small test subdomain G and look for the difference between the highest and the lowest mean temperature we can get on G taking into account admissible conductivities.

Since the body is anisotropic, the Kirchhoff transformation cannot be applied to get rid of the nonlinearity in the state equation. Also, cost functional gradient computation is more complex than in the case of an isotropic material (cf. [1]).

The paper is organized as follows. In Section 2, we briefly introduce the problem and its approximation, and give a survey of relevant existence as well as convergence

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results. Section 3 deals with sensitivity analysis, i.e., we focus on the gradient of the cost functional. Finally, numerical examples are presented in Section 4.

2. Maximal difference problem and its approximation

To give the reader an insight into a reliable solution concept and into the problem in question, we follow [2] except for a few alterations which, however, do not endanger validity of the cited statements. We will often make use of physical terminology to put the problem into a context of physics.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with a Lipschitz boundary $\partial\Omega$ composed of relatively open parts Γ_1 and Γ_2, $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$, $\mathrm{meas}_{d-1}(\overline{\Gamma}_1 \cap \overline{\Gamma}_2) = 0$.

We consider a diagonal $(d \times d)$ matrix $A(u)$ of heat conductivities $a_i(u)$, $i = 1, \dots, d$, dependent on the temperature u and belonging to the respective sets $\mathscr{U}^i_{\mathrm{ad}}$ defined later on.

The state boundary value problem reads

$$-\operatorname{div}(A(u)\ \mathrm{grad}\ u) = f(x,u) \quad \text{in } \Omega, \tag{2.1}$$

$$u = \bar{u} \quad \text{on } \Gamma_1, \tag{2.2}$$

$$n^{\mathrm{T}} A(u)\ \mathrm{grad}\ u + \alpha(s,u)u = g(s,u) \quad \text{on } \Gamma_2, \tag{2.3}$$

where $\bar{u}$ is a given function, n is the unit outward normal to $\partial\Omega$, and f, g, α are the bounded measurable functions. To specify them further, we define

$$U(C_L) = \{a \in C^{(0),1}(\mathbb{R}^1) \text{ (i.e., Lipschitz functions)}: |\mathrm{d}a/\mathrm{d}t| \le C_L \text{ a.e. in } \mathbb{R}^1\},$$

$$U_1 = \{a \in C^{(0),1}(\mathbb{R}^1): \forall t < T_{\mathrm{c}}\ a(t) = a(T_{\mathrm{c}}),\ \forall t > T^{\mathrm{c}}\ a(t) = a(T^{\mathrm{c}})\},$$

where C_L, T_{c}, T^{c} are the given constants, $-\infty < T_{\mathrm{c}} < T^{\mathrm{c}} < \infty$, $C_L > 0$. We assume that positive constants C_{Lf}, C_{Lg}, $C_{L\alpha}$ exist such that

$$f(x,\cdot) \in U(C_{Lf}),\ g(s,\cdot) \in U(C_{Lg}),\ \alpha(s,\cdot) \in U(C_{L\alpha}) \tag{2.4}$$

for almost all $x \in \Omega$ and $s \in \Gamma_2$, respectively. Next, let $\alpha(s,\xi) \ge 0$ for almost all $s \in \Gamma_2$ and all $\xi \in \mathbb{R}^1$.

The definition of the admissible heat conductivities should reflect the fact that, due to inaccurate measurements, their dependence on the temperature is not known exactly but only within certain limits. We suppose that each of the below-defined admissible sets $\mathscr{U}^i_{\mathrm{ad}}$, $i=1,\dots,d$, is determined by a governing function $\hat{a}_i$ and a slope deviation function $\hat{k}_i$. To this end, we introduce positive, piecewise continuously differentiable functions $\hat{a}_i \in U_1 \cap U(C_L)$ and nonnegative functions $\hat{k}_i$ continuous on $[T_{\mathrm{c}}, T^{\mathrm{c}}]$, $i = 1, \dots, d$.

Let μ_N be a uniform subdivision of $[T_{\mathrm{c}}, T^{\mathrm{c}}]$ into N segments determined by a set $p(\mu_N)$ of nodal points from $[T_{\mathrm{c}}, T^{\mathrm{c}}]$. Let us have an increasing sequence $\Lambda = \{N_j\}_{j=1}^{\infty}$ of integers $N_j \to \infty$.

For $i = 1, \dots, d$, we define

$\mathscr{U}^i_1 = \mathrm{cl}\ \{a \in U_1 : \exists N \in \Lambda\ \forall t \in p(\mu_N)\ |a'(t) - \hat{a}'_i(t)| \le \hat{k}_i(t)$ and a is piecewise linear on $\mu_N\}$,