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Engineering Analysis with Boundary Elements 25 (2001) 347–362

ENGINEERING ANALYSIS *with* BOUNDARY ELEMENTS

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Boundary element based formulations for crack shape sensitivity analysis☆

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Abstract

The present paper addresses several BIE-based or BIE-oriented formulations for sensitivity analysis of integral functionals with respect to the geometrical shape of a crack. Functionals defined in terms of integrals over the external boundary of a cracked body and involving the solution of a frequency-domain boundary-value elastodynamic problem are considered, but the ideas presented in this paper are applicable, with the appropriate modifications, to other kinds of linear field equations as well. Both direct differentiation and adjoint problem techniques are addressed, with recourse to either collocation or symmetric Galerkin BIE formulations. After a review of some basic concepts about shape sensitivity and material differentiation, the derivative integral equations for the elastodynamic crack problem are discussed in connection with both collocation and symmetric Galerkin BIE formulations. Building upon these results, the direct differentiation and the adjoint solution approaches are then developed. In particular, the adjoint solution approach is presented in three different forms compatible with boundary element method (BEM) analysis of crack problems, based on the discretized collocation BEM equations, the symmetric Galerkin BEM equations and the direct and adjoint stress intensity factors, respectively. The paper closes with a few comments.

Keywords: Boundary element method; Galerkin BIE formulations; Crack shape sensitivity; Direct differentiation; Adjoint state

1. Introduction

The consideration of sensitivity analysis of integral functionals with respect to shape parameters arises in many situations where (part of) a geometrical domain is either unknown or variable. Shape optimization and inverse problems are the most obvious instances, but not the only ones; for instance, the energy release rate, a basic concept of fracture mechanics, is mathematically defined as (minus) the derivative of the potential energy at equilibrium with respect to crack front perturbations. For these reasons, the numerical evaluation of sensitivities of functionals with respect to shape perturbations is clearly an important issue. The present paper is specifically concerned with boundary element-based methods for computing the sensitivity of integral functionals with respect to crack shape perturbations.

This goal is achievable by resorting to either finite-difference methods, considering small but finite domain perturbations, or analytical differentiation followed by discretization. The analytical approach is a priori clearly superior in terms of both accuracy and efficiency. It relies on either the adjoint variable approach or a direct differentiation of the field equations formulated in weak or BIE fashion. A substantial research effort has been devoted in the last decade or so to various formulations and applications of sensitivity analyses based on analytical differentiation with respect to shape parameters, or on the related mathematical concept of *domain derivative* [41,42]. As a result, these concepts are successfully applied to more and more engineering problems (see e.g. [21,29,30], among a quite abundant literature).

Further, since (the shape of) the boundary plays a key role in problems with variable or unknown domains, it is often found convenient, or even essential, to resort to the boundary element method (BEM). Both the adjoint problem [1,4,17–20,34] and the direct differentiation approach [2,5,23,32,33,36,37,39,45] have been investigated in connection with BEMs (see also the journal special issue [15]). Besides, defect identification problems are sometimes solved using successive linearizations of measurement residuals [24,25].

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☆ Submitted to Engineering Analysis with Boundary Elements, revised version.

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PII: S0955-7997(01)00025-X

crack. The functionals considered here are defined in terms of integrals over the external boundary involving the solution of a linear boundary-value problem in frequency-domain elastodynamics. Both direct differentiation and adjoint problem techniques are addressed, with recourse to either collocation or symmetric Galerkin BIE formulations. Following the statement of a generic direct elastodynamic problem (Section 2) and a review of some basic concepts about shape sensitivity and material differentiation (Section 3), integral identities in derivative form are established in Section 4. These results allow to formulate the derivative integral equations for the elastodynamic crack problem defined in Section 2 in connection with collocation BIE (Section 5) and symmetric Galerkin BIE (Section 6). Building upon these results, the direct differentiation and the adjoint solution approaches are discussed in Sections 7 and 8 respectively. In particular, the adjoint solution approach is presented in three different forms compatible with BEM analysis of crack problems. The paper closes with a few comments (Section 9).

2. The direct problem

Let us consider, in the three-dimensional Euclidean space $\mathbb{R}^3$ equipped with a Cartesian orthonormal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, an elastic body $\Omega \subset \mathbb{R}^3$ of finite extension, externally bounded by the closed surface S and containing a crack Γ. The unit normal $\mathbf{n}$ to Γ is oriented along the $\Gamma^- \to \Gamma^+$ direction, where Γ^+, Γ^- are the two crack faces (Fig. 1) (the outward normal to $\Gamma^\pm$ is thus $\mp\boldsymbol{n}$). The displacement $\boldsymbol{u}$, strain $\boldsymbol{\varepsilon}$ and stress $\boldsymbol{\sigma}$ in Ω are related by the field equations:

$$\text{div}\,\boldsymbol{\sigma} + \rho\omega^2\boldsymbol{u} = 0 \qquad \boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon} \qquad \boldsymbol{\varepsilon} = \tfrac{1}{2}(\nabla\boldsymbol{u} + \nabla^{\mathrm{T}}\boldsymbol{u}) \tag{1}$$

where $\mathbf{C}$ denotes the fourth-order elasticity tensor, given in the isotropic case by:

$$\mathbf{C}_{abcd} = \mu\left(\frac{2\nu}{1-2\nu}\delta_{ab}\delta_{cd} + \delta_{bd}\delta_{ac} + \delta_{bc}\delta_{ad}\right)$$

where μ and ν denote, respectively, the shear modulus and the Poisson ratio. Besides, displacements and tractions are prescribed on the portions S_u and $S_p = S \backslash S_u$ of S, while the crack surface Γ is stress-free:

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad (\text{on } S_u) \tag{2}$$

$$\boldsymbol{p} = \hat{\boldsymbol{p}} \quad (\text{on } S_p)$$

$$\mathbf{p} = \mathbf{0} \quad (\text{on } \Gamma)$$

where $\mathbf{p} \equiv \boldsymbol{\sigma}\cdot\mathbf{n}$ is the traction vector, defined in terms of the outward unit normal $\mathbf{n}$ to Ω. For a given location of the crack, the field Eqs. (1) and boundary conditions (2) define the *direct problem.*

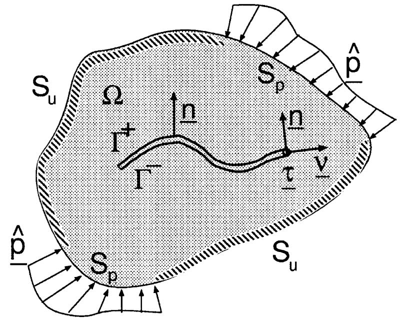

Fig. 1. Cracked elastic solid Ω and boundary conditions.

3. Material derivative in a shape perturbation

To investigate the effect of crack shape perturbations, the shape of the body Ω is assumed to depend on a parameter t (a fictitious, non-physical 'time') through a continuum kinematics-type Lagrangian description. The unperturbed, 'initial' configuration Ω is conventionally associated with $t = 0$:

$$\boldsymbol{x} \in \Omega \to \boldsymbol{x}^t = \boldsymbol{\Phi}(\boldsymbol{x}, t) \in \Omega(t) \qquad \boldsymbol{\Phi}(\boldsymbol{x}, 0) = \boldsymbol{x} \tag{3}$$

All 'time' derivatives will be implicitly taken at $t = 0$, i.e. the first-order effect of infinitesimal perturbations of $\Omega \equiv \Omega(0)$ is considered. The *geometrical transformation* $\boldsymbol{\Phi}(\cdot; t)$ must possess a strictly positive Jacobian for $t \geq 0$. A given domain evolution considered as a whole admits infinitely many different representations (3).

3.1. Material derivative of scalar or tensor fields

Differentiation of field variables and integrals in a domain perturbation is a well-documented subject, see e.g. Petryk and Mroz [40], Sokolowski and Olesio [42]; a few basic concepts and results are recalled now. The *initial transformation velocity* $\boldsymbol{\theta}$ is defined by:

$$\boldsymbol{\theta}(\boldsymbol{x}) = \frac{\partial \boldsymbol{\Phi}}{\partial t}\Big|_{t=0} \tag{4}$$

The 'material' (or 'Lagrangian') derivative at $t = 0$ of a field quantity $f(\boldsymbol{x}, t)$ in a geometrical transformation, denoted by $\overset{*}{f}$, is defined by:

$$\overset{*}{f} = \lim_{t \searrow 0} \frac{1}{t}[f(\boldsymbol{x}^t, t) - f(\boldsymbol{x}, 0)] = \frac{\partial f}{\partial t} + \boldsymbol{\nabla} f \cdot \boldsymbol{\theta} \tag{5}$$

The material derivative of the gradient of a field quantity is given by:

$$(\boldsymbol{\nabla} f)^* = \boldsymbol{\nabla}\overset{*}{f} - \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla}\boldsymbol{\theta} \tag{6}$$

3.2. Material derivative of surface integrals

The material derivatives of the unit normal $\mathbf{n}$ and the surface differential element $\mathrm{d}S$ on a material surface