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# Shape design sensitivity analysis of 2D anisotropic structures using the boundary element method

A. Tafreshi*

*Aerospace Engineering, The Manchester School of Engineering, The University of Manchester, Simon Building, Oxford Road, Manchester M13 9PL, UK*

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**Abstract**

A directly differentiated form of boundary integral equation with respect to geometric design variables is used to calculate shape design sensitivities for anisotropic materials. An optimum shape design algorithm in two dimensions is developed by the coupling of an optimising technique and a boundary element stress analyser for stress minimisation of anisotropic structures. Applications of this general-purpose program to the optimum shape design of bars and holes in plates with anisotropic material properties are presented. 

*Keywords*: Shape optimisation; Boundary element method; Design sensitivity analysis; Anisotropic materials

## 1. Introduction

In the previous study [1–4], numerical optimisation techniques and the boundary element method were combined to optimise the shape of 2D isotropic linear elastic structures subjected to static loading. The steps that were required were as follows: shape representation, boundary element analysis to calculate stresses and displacements, design sensitivity analysis for calculating derivatives, numerical optimisation to find the optimum solution iteratively, and boundary element mesh re-generation as the optimisation proceeds.

For shape representation Hermitian cubic spline functions were employed. The Hermitian cubic spline has two continuous derivatives everywhere, minimum mean curvature and is axis independent. It posses globally controlled properties, such that moving any design variable point on the curve will change the shape globally rather than just locally. Complex geometries can be modelled by a small set of design variables.

Design sensitivity analysis to calculate derivatives of displacements and stresses with respect to the design variables were carried out by implicit differentiation of the corresponding boundary element elasticity kernels. For verification purposes, the derivatives of displacements and stresses were calculated both by this direct analytical differentiation method and by the finite difference method. Not surprisingly, results obtained by analytical differentiation were much more accurate.

Two main types of problems were considered. Firstly, those problems involving the minimisation of the deviation of the von Mises equivalent stresses on the boundary of the region of interest from a desired uniform mean stress. In this case the objective function is highly non-linear, while the constraints are linear. The optimisation method used was the extended interior penalty function approach combined with the BFGS method [5] for unconstrained minimisation. The second main type of problem was the minimisation of structural weight, while satisfying certain constraints upon stresses and geometry. Since both the objective function and constraints are non-linear, the feasible direction method was employed.

In the present study, using a similar procedure, a general-purpose computer program for shape optimal design of 2D anisotropic structures in order to smooth stress peaks is presented.

The developed optimisation programme has three main components: an optimiser, a stress analyser and a design sensitivity analyser. Therefore, the main differences between the new programme and the one used for the isotropic materials [1–4] are in the stress analyser and design sensitivity analyser.

The analytical formulation of the direct boundary integral equation (BIE) for plane anisotropic elasticity may be well developed by following the same steps as in the isotropic

* Tel.: +44-(0)161-275-4459; fax: +44-(0)161-275-4261.
*E-mail address:* atafreshi@man.ac.uk (A. Tafreshi).

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**Nomenclature**

$A_{jk}$ complex constants
$C_{jk}(P)$ the limiting value of the surface integral of $T_{jk}(p,Q)$
$D_s$ $(s = 1,4)$ operator
$E_k$ young's modulus in the $x_k$ direction
$F$ objective function
$F_0$ objective function at initial step
$G_{12}$ shear modulus
$J(\xi)$ Jacobian of transformation from global Cartesian coordinates to intrinsic coordinates of the element
$m_{1k}$, $m_{2k}$ unit vectors tangent and normal to the surface
$n_1$, $n_2$ direction cosines of the unit outward normal vector to the surface of the elastic body
$N^c(\xi)$ quadratic shape function corresponding to the $c$th node of the element
$P$ load point at the surface of the elastic domain
$Q$ field point at the surface of the elastic domain
$(R_i,\theta_i)$ polar coordinates
$r_{ij}$ complex constants
$S_{\max}$ maximum equivalent stress
$S_{mn}$ elastic compliance matrix
$\bar{S}_{jk}$ transformed lamina compliance matrix
$t_j$ traction vector
$T_{jk}(P,Q)$ the $j$th component of the traction vector at point $Q$ due to a unit point load in the $k$th direction at $P$
$u_j$ displacement vector
$U_{jk}$ the $j$th component of the displacement vector at point $Q$ due to a unit point load in the $k$th direction at $P$
$V_1,V_2,V_3,V_4$ invariants
$x_i$ rectangular Cartesian coordinates
$z_j$ complex coordinates
$\alpha_j$, $\beta_j$ real constants
$\delta_{jk}$ Kronecker delta
$\epsilon_{jk}$ strain tensor
$\phi$ Airy stress function
$\gamma$ Lamina orientation angle with respect to the $xy$ axis
$\Lambda_1$, $\Lambda_2$ real functions of the Cartesian and intrinsic coordinates at each integration point
$\sigma_{L1}$, $\sigma_{L2}$ components of the applied load
$\sigma_{jk}$ stress tensor
$\sigma_e$ equivalent stress
$\zeta_i$ coordinates of load point
$\nu_{jk}$ Poisson's ratio
$\mu_s$ roots of the characteristic equation
$\Omega_1$, $\Omega_2$ real functions of the Cartesian and intrinsic coordinates at each integration point
$\zeta$ intrinsic coordinates of isoparametric quadratic element

case. For the details of these, the reader is referred to the references [6–11].

Here the implicit differentiation of the BIE for 2D anisotropic linear elastic materials is carried out and then stress and displacement derivatives are calculated. The accuracy is compared against the results of the finite difference applied to the boundary element analysis. Applications of the program to the optimum shape design of an infinite plate with a central circular hole under a biaxial stress field and a cantilever beam under uniform shear load at one end are presented. In each case isotropic and anisotropic materials are employed and the results are compared.

## 2. Constitutive equations for plane anisotropic elasticity

The stress–strain relations for a 2D homogeneous, anisotropic elastic body in plane stress is

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \quad (1)$$

where $\sigma_{jk}$ and $\epsilon_{jk}$ $(j,k = 1,2)$, are the stresses and strains, respectively, and the coefficients $S_{mn}$ are the elastic compliances of the material. These compliances can be