PERGAMON

International Journal of Solids and Structures 39 (2002) 2365–2385

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# Sensitivity analysis for shape perturbation of cavity or internal crack using BIE and adjoint variable approach

M. Bonnet [a,*], T. Burczyński [b], M. Nowakowski [b]

[a] *CNRS URA 7649, Laboratoire de Mecanique des Solides, Ecole Polytechnique, 91128 Palaiseau Cedex, France*

[b] *Department for Strength of Materials and Computational Mechanics, Silesian University of Technology, Konarskiego 18a, 44-100 Gliwice, Poland*

Received 26 September 2001; received in revised form 11 January 2002

**Abstract**

This paper deals with the application of the adjoint variable approach to sensitivity analysis of objective functions used for defect detection from knowledge of supplementary boundary data, in connection with the use of BIE/BEM formulations for the relevant forward problem. The main objective is to establish expressions for crack shape sensitivity, based on the adjoint variable approach, that are suitable for BEM implementation.

In order to do so, it is useful to consider first the case of a cavity defect, for which such boundary-only sensitivity expressions are obtained for general initial geometry and shape perturbations. The analysis made in the cavity defect case is then seen to break down in the limiting case of a crack. However, a closer analysis reveals that sensitivity formulas suitable for BEM implementation can still be established. First, particular sensitivity formulas are obtained for special shape transformations (translation, rotation or expansion of the crack) for either two- or three-dimensional geometries which, except for the case of crack expansion together with dynamical governing equations, are made only of surface integrals (three-dimensional geometries) or line integrals (two-dimensional geometries). Next, arbitrary shape transformations are accommodated by using an additive decomposition of the transformation velocity over a tubular neighbourhood of the crack front, which leads to sensitivity formulas. This leads to sensitivity formulas involving integrals on the crack, the tubular neighbourhood and its boundary. Finally, the limiting case of the latter results when the tubular neighbourhood shrinks around the crack front is shown to yield a sensitivity formula involving the stress intensity factors of both the forward and the adjoint solutions. Classical path-independent integrals are recovered as special cases.

The main exposition is done in connection with the scalar transient wave equation. The results are then extended to the linear time-domain elastodynamics framework. Linear static governing equations are contained as obvious special cases. Numerical results for crack shape sensitivity computation are presented for two-dimensional time-domain elastodynamics. © 2002 Elsevier Science Ltd. All rights reserved.

*Keywords:* Geometrical inverse problem; Wave propagation; Elastodynamics; Adjoint variable approach; Material derivative; Defect identification; Boundary integral equations; Boundary element method

* Corresponding author. Tel.: +33-1-6933-3327/4129; fax: +33-1-6933-3026.
*E-mail addresses:* bonnet@lms.polytechnique.fr (M. Bonnet), burczyns@rmt4.kmt.polsl.gliwice.pl (T. Burczyński).

0020-7683/02/$ - see front matter 
PII: S0020-7683(02)00131-2

## 1. Introduction

The consideration of sensitivity analysis of integral functionals with respect to shape parameters arises in many situations where a geometrical domain plays a primary role; shape optimization and inverse problems are the most obvious, as well as possibly the most important, of such instances.

It is well known that, apart from resorting to approximative techniques such as finite differences, shape sensitivity evaluation can be dealt with using either the direct differentiation approach or the adjoint variable approach (see, e.g. Burczyński, 1993b), the present paper being focused on the latter. Besides, consideration of shape changes in otherwise (i.e. for fixed shape) linear problems makes it very attractive to use boundary integral equation (BIE) formulations, which constitute the minimal modelling as far as the geometrical support of unknown field variables is concerned.

In the BIE context, the direct differentiation approach rests primarily upon the material differentiation of the governing integral equations. This step has been studied by many researchers, from BIE formulation in either singular form (Barone and Yang, 1989; Mellings and Aliabadi, 1995) or regularized form (Bonnet, 1995b; Matsumoto et al., 1993; Nishimura et al., 1992; Nishimura, 1995). Following this approach, the process of sensitivity computation needs the solution of as many new boundary-value problems as the numbers of shape parameters present. The fact that they all involve the same, original, governing operator reduces the computational effort to the building of new right-hand sides and the solution of linear systems by backsubstitution. The usual material differentiation formula for surface integrals is shown in Bonnet (1997) to be still valid when applied to strongly singular or hypersingular formulations. Thus, the direct differentiation approach is in particular applicable in the presence of cracks.

The adjoint variable approach is even more attractive, since it requires the solution of only one new boundary-value problem (the so-called adjoint problem) per integral functional present (often only one), whatever the number of shape parameters. In connection with BIE formulations alone, the adjoint variable approach has been successfully applied to many shape sensitivity problems (see, e.g. Aithal and Saigal, 1995; Bonnet, 1995a; Burczyński, 1993a; Burczyński and Fedelinski, 1992; Burczyński et al., 1995; Choi and Kwak, 1988; Meric, 1995). This relies heavily upon the possibility of formulating the final, analytical expression of the shape sensitivity of a given integral functional as a boundary integral that involves the values taken by the primary and adjoint states on the boundary. However, obtaining this boundary-only expression raises mathematical difficulties when the geometrical domain under consideration contains cracks or other geometrical singularities; non-integrable terms associated with, e.g. crack tip singularity of field variables appear in some expressions.

This paper deals with the formulation of the adjoint variable method applied to sensitivity analysis, in connection with the use of BIE formulations for the transient wave equation. Typical problems where this approach is useful are inverse problems of cavity or crack detection from transient wave measurements on a part of the external boundary, where the integral functionals considered express the gap between measured and computed data on the external boundary, e.g. in the form of a least-squares distance. However, the sensitivity results are derived for more general boundary integral functionals. The formulation of the adjoint problem and the corresponding boundary-only formula for the shape sensitivity of the functional are established for the case of an unknown cavity. The latter is then shown to become inconsistent in the limit when the cavity becomes a crack, due to the non-integrability of a certain domain integral, causing an integration-by-parts process to break down. However, resting on the analysis made for the case of a cavity, functional shape sensitivity expressions consistent with the use of BIE formulations and applicable to crack identification problems are derived in three different forms. Firstly, simple shape transformations (translations, rotations, expansion) are considered. Secondly, a sensitivity formula involving integrals on the crack, on an arbitrary tubular neighbourhood of the crack front and on its boundary is derived. Thirdly, the limiting case of the latter result when the tubular neighbourhood shrinks around the crack front is shown to yield a sensitivity formula involving the stress intensity factors of both the forward and the