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Mixed Equilibrium Problems: Sensitivity Analysis and Algorithmic Aspect

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Abstract—The aim of this paper is twofold. First, it is to extend the sensitivity analysis framework, developed recently for variational inequalities, to mixed equilibrium problems. The second is to propose iterative methods for solving this kind of problems. In the process, we establish an equivalence between an extended version of Wiener-Hopf equations and the given problems relying on a generalization of the Yosida approximation notion. Our results generalize results obtained for optimization, variational inequalities, complementarity problems, and problems of Nash equilibria.

Keywords—Equilibrium problems, Sensitivity, Wiener-Hopf equations, Proximal methods.

1. INTRODUCTION AND PRELIMINARIES

Equilibrium problems theory has emerged as an interesting and fascinating branch of applicable mathematics. This theory has become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization, operation research in a general and unified way. There is a substantial number of papers on existence results for solving equilibrium problems based on different relaxed monotonicity notions and various compactness assumptions. But up to now no sensitivity analysis and only few iterative methods to solve such problems have been done. Inspired and motivated by the recent research developed for variational inequalities, we consider a class of mixed equilibrium problems which includes variational inequalities as well as complementarity problems, convex optimization, saddle point-problems, problems of finding a zero of a maximal monotone operator, and Nash equilibria problems as special cases. Using Wiener Hopf equations technique and adapting ideas of Dafermos [1] and Noor [2], we develop, in the first part of this paper, a sensitivity analysis relying on a fixed point formulation of the given problem. This formulation is obtained thanks to a generalization of the Yosida approximation notion introduced in [3] and allows us to derive, in the second part of this paper, iterative methods for such problems.

Let X be a real Hilbert space and $\|.\|$ the norm generated by the scalar product $\langle .,. \rangle$. Consider the problem of finding

$$\bar{x} \in K; \quad F(g(\bar{x}), y) + \langle T\bar{x}, y - \bar{x} \rangle \geq 0, \qquad \forall y \in K, \tag{MEP}$$

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where K is a nonempty, convex, and closed set of X, T, $g : K \to K$ are two nonlinear operators and $F : K \times K \to \mathbb{R}$ is a given bifunction satisfying $F(x,x) = 0$ for all $x \in K$.

This problem has potential and useful applications in nonlinear analysis and mathematical economics. For example, if we set $F(x,y) = \varphi(y) - \varphi(x)$, for all $x, y \in K$, $\varphi : K \to \mathbb{R}$ a real-valued function, $g = I$ and $T = 0$, then (MEP) reduces to the following minimization problem subject to implicit constraints:

$$\text{find } \bar{x} \in K \text{ such that } \varphi(\bar{x}) \leq \varphi(y), \qquad \forall y \in K. \tag{CO}$$

The basic case of variational inclusions corresponds to $F(x,y) = \sup_{\zeta \in Bx} \langle \zeta, y - x \rangle$ with $B : K \rightrightarrows K$ a set-valued maximal monotone operator. Actually, the mixed equilibrium problem (MEP) is nothing but

$$\text{find } \bar{x} \in K \text{ such that } 0 \in T(\bar{x}) + B(g(\bar{x})), \qquad \forall y \in K. \tag{VP}$$

Moreover, if $F(x,y) = \varphi(y) - \varphi(x)$, then inclusion (VP) reduces to

$$\text{find } \bar{x} \in K \text{ such that } \varphi(y) - \varphi(x) + \langle T(\bar{x}), y - \bar{x} \rangle \geq 0, \qquad \forall y \in K. \tag{VI}$$

In particular if $\varphi = 0$ and K is a closed convex cone, then inequalities (VI) can be written as

$$\text{find } \bar{x} \in K; \qquad T(\bar{x}) \in K^* \quad \text{and} \quad \langle T(\bar{x}), \bar{x} \rangle = 0, \tag{CP}$$

where $K^* = \{x \in X : \langle x, y \rangle \geq 0, \ \forall y \in K\}$ is the polar cone to K. The problem of finding such a $\bar{x}$ is an important instance of the well-known complementarity problem of mathematical programming.

Another example corresponds to Nash equilibria in noncooperative games. Let I (the set of players) be a finite index set. For every $i \in I$ let K_i (the strategy set of the i^{th} player) be a given set, f_i (the loss function of the i^{th} player, depending on the strategies of all players): $K \to \mathbb{R}$ a given function with $K := \prod_{i \in I} K_i$. For $x = (x_i)_{i \in I} \in K$, we define $x^i := (x_j)_{j \in I,\, j \neq i}$. The point $\bar{x} = (\bar{x}_i)_{i \in I} \in K$ is called a Nash equilibrium if and only if for all $i \in I$ the following inequalities hold true:

$$f_i(\bar{x}) \leq f_i(\bar{x}^i, y_i), \qquad \text{for all } y_i \in K_i, \tag{NE}$$

(i.e., no player can reduce his loss by varying his strategy alone). Let $g = I$ and $T = 0$ and define $F : K \times K \to \mathbb{R}$ by

$$F(x,y) = \sum_{i \in I} \left(f_i(x^i, y_i) - f_i(x) \right).$$

Then $\bar{x} \in K$ is a Nash equilibrium if, and only if, $\bar{x}$ solves (MEP). The following definitions and theorem will be needed in the sequel (see for example [4]).

DEFINITION 1. *Let $F : K \times K \to \mathbb{R}$ be a real valued bifunction.*

(i) *F is said to be monotone, if*

$$F(x,y) + F(y,x) \leq 0, \qquad \text{for each } x, y \in K. \tag{1}$$

(ii) *F is said to be strictly monotone if*

$$F(x,y) + F(y,x) < 0, \qquad \text{for each } x, y \in K, \text{ with } x \neq y. \tag{2}$$

(iii) *F is upper-hemicontinuous, if for all x, y, $z \in K$*

$$\limsup_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x,y). \tag{3}$$