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SENSITIVITY ANALYSIS OF MECHANICAL SYSTEMS WITH UNILATERAL CONSTRAINTS†

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The problem of sensitivity analyses for mechanical systems with unilateral constraints is considered. The problem is formulated as one of computing the derivative, with respect to the vector of parameters, of a functional characterizing the motion. To compute the derivative, the adjoint variable approach is extended to systems with unilateral constraints. The set of active constraints may change in the course of the motion, with or without impact. Jump conditions for the adjoint variables are indicated for the times at which changes occur in the set of active constraints. As an example, a mechanical system whose motion is constrained by an absolutely elastic stop is considered.

Sensitivity analysis is used in problems which arise when it is desired to assess the influence of measurement errors or inaccuracies in estimates of external forces acting on real systems [1–3]. There are also applications of sensitivity analysis to projection problems, modelled either by systems of ordinary differential equations or by partial differential equations [1, 2]. However, these applications are concerned mainly with problems without constraints or with bilateral constraints. At the same time, there is a large class of systems with unilateral constraints for which it seems interesting to use sensitivity analysis. The monographs [4, 5], which are devoted to systems with unilateral constraints, present examples of vibro-elastic systems, systems of cyclic automation [4], and systems that arise when solving problems in the dynamics of structures [5]. The most common sensitivity measure is the derivative with respect to a parameter of some functional that can be computed along the trajectories of motion of the system [3, 6]. General approaches to the computation of such derivatives along trajectories of dynamical systems are known [7, 8]. In this paper formulae are derived for the derivative of an integral functional which characterizes the sensitivity of a mechanical system with unilateral constraints in the case of the impact of the system against one constraint or release from the constraint.

1. EQUATIONS OF MOTION

Consider a system of ordinary differential equations describing the motion of a mechanical system with ideal unilateral holonomic constraints, confining our attention to finite inequality constraints

$$M\ddot{q} = F(q, \dot{q}, t) + G\lambda, \quad \Phi(q) \geq 0, \quad G = \partial\Phi(q)/\partial q \in R^{n\times m}, \quad t \in [t_1, T] \tag{1.1}$$

where $q \in R^n$ is the vector of generalized coordinates, $M \in R^{n\times n}$ is the symmetric positive-definite matrix of the masses, $F(q,\dot{q}, t)$ is the vector of forces, $\Phi: R^n \to R^m_+$ is a continuously differentiable vector function whose components describe holonomic stationary unilateral constraints and $\lambda(t)$ is the vector of Lagrange multipliers.

The Lagrange multiplier λ_i corresponding to the ith constraint satisfies the following complementarity condition [9, 10].

$$\lambda_i \geq 0, \quad \Phi_i \geq 0, \quad \lambda_i\Phi_i = 0 \tag{1.2}$$

We shall consider system (1.1) in the interval $[t_1, T]$ as a composite or a system with variable structure. To that end, the whole interval is divided into subintervals

$$[t_j, t_{j+1}], \quad j = 1, 2, \ldots, N-1, \quad t_N = T \tag{1.3}$$

within each of which the set of active constraints does not change. The union of these subintervals is the entire time interval $[t_1, T]$.

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To compute the trajectories of motion of the mechanical system, the correspondence between the sets of active constraints and the time subintervals (1.3) must be determined, and the conditions governing the relation between the values of the generalized velocities at the end-points of the subintervals (transition with or without impact) must be determined. To determine the equations of motion of system (1.1), that is, to indicate the set of active constraints in each subinterval, one generally uses a generalization of Gauss's principle of least constraint [11] to systems with unilateral constraints [12], or considers the mechanical system as a system satisfying a complementarity condition [9, 10].

Gauss's principle of least constraint leads to the optimization problem

$$\min_{\ddot{q}} \frac{1}{2}(\ddot{q} - M^{-1}F)^T M(\ddot{q} - M^{-1}F) \tag{1.4}$$

under constraints

$$G^T\ddot{q} + \dot{G}^T\dot{q} \geq 0 \tag{1.5}$$

The solution of this quadratic programming problem with linear constraints satisfies the Kuhn–Tucker Theorem [13], whose geometrical meaning is that, for an optimal solution, the antigradient of a suitable minimization of the expression (1.4) may be expressed as a non-negative linear combination ($\lambda_i \geq 0$) with the sign of the columns of the matrix G corresponding to active constraints reversed. Consequently, the matrix G corresponding to the interval (1.3) may be obtained from the Kuhn–Tucker conditions. The Kuhn–Tucker system for a quadratic programming problem may be reduced to a linear complementarity problem. Consequently, problem (1.4), (1.5) may be solved by using the procedures of [12, 13]. Complementarity problems are widely used to describe mechanical systems with unilateral constraints [10, 12, 14].

After determining the set of columns of the matrix G corresponding to a subinterval (1.3), it is convenient for the integration of Eqs (1.1) to exclude the multiplier λ [15].

We will now consider the relation between the values of the generalized velocities of the mechanical system at the common end-point of two adjacent subintervals. The transition from one subinterval of the time of motion to another may be of either of two types: with or without impact. To obtaining defining relations in impact theory one most frequently makes use of Newton's hypothesis [4, 16]: the change in the normal component of the generalized velocity of colliding bodies depends only on their materials, not on the velocities, and satisfies the relation [4, 16]

$$g^T\dot{q}(t^+) = -eg^T\dot{q}(t^-) \tag{1.6}$$

where e is the coefficient of restitution of the velocity on impact, which varies from zero to unity, and g is the gradient of the equation of the constraint.

The change in velocities on impact at time t_k against a constraint satisfying condition (1.6) may be expressed in the form

$$\dot{q}(t_k^+) = \dot{q}(t_k^-) - (1+e)\frac{g^T\dot{q}(t_k^-)}{g^T M^{-1} g} M^{-1} g \tag{1.7}$$

In the case of absolutely elastic impact ($e = 1$) this formula, for scleronomous systems with one unilateral constraint, is identical with formula (13.15) in [4].

Thus, the system considered in this paper may be classified as a mechanical system with unilateral constraints – inequality constraints on the generalized coordinates. At the impact times, the generalized coordinates of the system are continuous, but the generalized velocities experience jumps. When the generalized velocities change as a result of impact, their initial values for the appropriate subinterval (1.3) are evaluated using formula (1.7), depending on the assumptions adopted concerning the value of the coefficient $e = 1$. Within each subinterval (1.3) the system satisfies equations of motion of the form (1.1).

2. SENSITIVITY ANALYSIS

Sensitivity analysis will be carried out below for the case in which the mechanical system impacts one constraint and the case in which it is released from the constraint. We shall assume that the equations of motion of the system have been reduced to a first-order system of ordinary differential equations